Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations Sub-structure and Hahn series

fields.

Computations with surreal numbers, existing

Cauchy completion Computable Analysis over surreal fields

integration Motivation : Analog computing Some previous tries Handle the gaps, a new notion of compacity.

Strongly contin functions Surreal Numbers, integration and computations

Quentin Guilmant

LIX, École Polytechnique

26 novembre 2019

Surreal numbers 97-11-6102

Surreal Numbers, integration and computations

Quentin Guilmant

LIX, École Polytechnique

26 novembre 2019

Quentin Guilmant

ntroduction : Numbers

Surreal Numbers

Definitions and operations Sub-structure and

Hahn series Gaps in the surreal fields.

Computations with surreal numbers, existing methods Cauchy completion Computable Analysis over surreal fields

Problem of integration Motivation : Analog computing Some previous tries Handle the gaps, a new notion of compacity. Strongly continuous functions

Perspectives

1 Introduction : Numbers

2 Surreal Numbers

Definitions and operations Sub-structure and Hahn series Gaps in the surreal fields.

3 Computations with surreal numbers, existing methods Cauchy completion Computable Analysis over surreal fields

4 Problem of integration

Motivation : Analog computing Some previous tries Handle the gaps, a new notion of compacity. Strongly continuous functions Perspectives Surreal numbers 97-11-50 6105 Introduction : Numbers

Surreal Numbers Definitions and operations Sub-structure and Hahn series Gaps in the surreal fields.

Computations with surreal numbers, existing methods Cauchy completion Computable Analysis over surreal fields

O Problem of integration

Motivation : Analog computing Some previous tries Hardle the gaps, a new notion of compacity. Strongly continuous functions Perspectives

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations

Sub-structure and Hahn series Gaps in the surreal fields.

1 Introduction : Numbers

with surreal numbers, existing

Cauchy completion Computable Analys

Problem of integration

Motivation : Analog computing

Some previous tries

Handle the gaps, new notion of compacity.

Strongly continuou functions

Perspectives

Outline

Introduction : Numbers

Outline

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations Sub-structure and

Gaps in the surreal fields.

with surreal numbers, existing methods Cauchy completion Computable Analysis over surreal fields

Problem of integration Motivation : Ana computing

Some previous tries

Handle the gaps, new notion of compacity.

Strongly continuou functions

Perspective

Numbers

- Construction of numbers : $\varnothing \to \mathbb{N} \to \mathbb{Q} \to \mathbb{R}$
- \mathbb{R} is the unique Archimedean and complete field.

Surreal numbers P7-II-6100 Numbers

Construction of numbers : Ø → N → Q → R
 R is the unique Archimedean and complete field

- If we want to talk about surreal numbers, the very first thing to wonder what is a number. In classical set theory we start from the emptyset and build numbers, then rationnal numbers and finally with Cauchy-completion or Dedekind-completion.
- \mathbb{R} is the unique field that is both Archimedean and complete. This properties and fundamental to prove fundamental theorems of analysis (TVE, TVI, Rolle, Mean Value Theorem, Fixed Point Theorems).

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

- Definitions and operations Sub-structure and
- Gaps in the surreal fields.
- Computations with surreal numbers, existing methods Cauchy completion Computable Analysis

Problem of integration Motivation : Anal computing Some previous tri-Handle the gaps, armostic of

- compacity. Strongly continuo
- Perspect

Numbers

• Construction of numbers : $\emptyset \to \mathbb{N} \to \mathbb{O} \to \mathbb{R}$

• \mathbb{R} is the unique Archimedean and complete field.

• What about replace \mathbb{N} by a set of ordinal number?

Normal form : $\sum n_i \omega^{\alpha_i}$

Get surreal numbers.

 $i < \alpha$

Surreal numbers Introduction : Numbers Numbers

-26

÷.

-

2019-1

 Construction of numbers: D → N → Q → R
 R is the unique Archimedean and complete field.
 What about replace N by a set of ordinal number? Normal form : ∑_{i < n}nⁿⁱⁿ Get sureal numbers.

- If we want to talk about surreal numbers, the very first thing to wonder what is a number. In classical set theory we start from the emptyset and build numbers, then rationnal numbers and finally with Cauchy-completion or Dedekind-completion.
- R is the unique field that is both Archimedean and complete. This properties and fundamental to prove fundamental theorems of analysis (TVE, TVI, Rolle, Mean Value Theorem, Fixed Point Theorems).
- With ordinal number instead of natural numbers we may get other things. We may get a lot of new numbers, that may look like the normal form of ordinal theorems.

Quentin Guilmant

Introduction Numbers

Surreal Numbers

Definitions and operations

Sub-structure and Hahn series Gaps in the surreal fields.

2 Surreal Numbers

Computation

with surreal numbers, existing methods Cauchy completi

Computable Analysis over surreal fields

Problem of integration

Motivation : Analog computing

Some previous tries

Handle the gaps, new notion of compacity.

Strongly continuou functions

Perspectives



Surreal numbers ⁹⁷ – Surreal Numbers ¹⁰ – Outline

O Surreal Numbers

Outline

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations

Sub-structure and Hahn series Gaps in the surreal

Computations with surreal numbers, existing methods Cauchy completion Computable Analys

Problem of integration

Motivation : Analo, computing

Some previous tries

Handle the gaps, new notion of compacity.

Strongly continuous functions

Perspective

3 methods to think about surreal numbers

 $\frac{\omega}{2} = [n \in \mathbb{N} \mid \{\omega - n\}]$

Cuts (and games) : *à la* Conway

Surreal numbers Surreal Numbers Definitions and operations -3 methods to think about surreal numbers

There are 3 main methods to define the surael numbers.

• The first is the vision of Conway. Conway was thinking them as cuts : basically they are games that have strategies for Alice or Bob.

3 methods to think about surreal numbers

• Cuts (and games) : à la Conway $\underline{w} = [n \in \mathbb{N} \mid \{\omega - n\}]$

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

> Definitions and operations

Sub-structure and Hahn series Gaps in the surreal

Computations

with surreal numbers, existing methods Cauchy completion

Computable Analysis over surreal fields

Problem of integration Motivation : Anal

Some previous tries

Handle the gaps, new notion of compacity.

Strongly continuou functions

Perspectives

3 methods to think about surreal numbers

Cuts (and games) : *à la* Conway Sign expansion : *à la* Gonshor $\frac{\omega}{2} = [n \in \mathbb{N} \mid \{\omega - n\} \\ \frac{\omega}{2} = (+)^{\omega} (-)$

Surreal numbers Surreal Numbers Definitions and operations -1 -1 -1 -2 -1 -3 methods to think about surreal numbers 3 methods to think about surreal numbers

 Cuts (and games) : à la Conway ¹/₂ = [n ∈ N | {ω − n²}
 Sign expansion : à la Gonshor
 ¹/₂ = (+)^ω(−
) ¹/₂ = (+)^ω(−
) ¹

There are 3 main methods to define the surael numbers.

- The first is the vision of Conway. Conway was thinking them as cuts : basically they are games that have strategies for Alice or Bob.
- Gonshor give another vision that is very connected with Conway's view. It introduce the sign expansion. Length of this sequence correspond to the birthday of the numbers in Conway's view.

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations

Sub-structure and Hahn series Gaps in the surreal

Computations with surreal numbers, existing methods Cauchy completion Computable Analysis over surreal fields

Problem of integration Motivation : Ana computing Some previous tri

Handle the gaps, new notion of compacity.

Strongly continuou functions

3 methods to think about surreal numbers

 $\frac{\omega}{2} = [n \in \mathbb{N} \mid \{\omega - n\}]$

 $\frac{\omega}{2} = (+)^{\omega}(-)$

 $\sum_{i<\lambda}r_i\omega$

- Cuts (and games) : *à la* Conway
- Sign expansion : *à la* Gonshor
- Hahn series : à la computer algebra

3 methods to think about surreal numbers

Cuts (and games): à la Conway ¹/₂ = [n ∈ N | {
 Sign expansion : à la Gonshor ¹/₂ = (+
 Hahn series : à la computer algebra

There are 3 main methods to define the surael numbers.

- The first is the vision of Conway. Conway was thinking them as cuts : basically they are games that have strategies for Alice or Bob.
- Gonshor give another vision that is very connected with Conway's view. It introduce the sign expansion. Length of this sequence correspond to the birthday of the numbers in Conway's view.
- After defining operations (×, +, a → ω^a) we can define expansion with series. Gonshor has made a connection with the sign expansion that is "natural".

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations

Sub-structure and Hahn series Gaps in the surreal

Computations with surreal numbers, existing methods Cauchy completion Computable Analysis over surreal fields

ntegration Motivation : Analog computing Some previous tries Handle the gaps, a new notion of compacity. Strongly continuous

Perspectiv

3 methods to think about surreal numbers

 $\frac{\omega}{2} = [n \in \mathbb{N} \mid \{\omega - n\}]$

 $\frac{\omega}{2} = (+)^{\omega}(-)$

 $\sum_{i<\lambda}r_i\omega$

- Cuts (and games) : *à la* Conway
- Sign expansion : à la Gonshor
- Hahn series : *à la* computer algebra

No is the <u>class</u> of all surreal numbers. It contains \mathbb{R} , the ordinals and many (many (many)) other ones ($\underline{\mathsf{Ex}} : \frac{\omega}{2}, \sqrt{\omega}, \omega^{1/\omega} = \log \omega$).

Surreal numbers Surreal Numbers Definitions and operations -3 methods to think about surreal numbers

• Cuts (and games) : 3 is Conscoy $\frac{W}{2} = \left[n \in \mathbb{N} \mid \left\{\omega - \right. \\ Sign expansion : 3 is Gonshor <math>\frac{W}{2} = \left\{+\right\}^n$ • Hahn series : 3 is composer algebra $\frac{W}{2}$. We then the series is the form the series of the series of the series \mathbb{N} to be the gives of all series all matters. It contains \mathbb{R} , the

3 methods to think about surrea

numbers

ordinals and many (many (many)) other ones (Ex : \tilde{g} , $\sqrt{\omega}$ $\omega^{1/\omega} = \log \omega$).

There are 3 main methods to define the surael numbers.

- The first is the vision of Conway. Conway was thinking them as cuts : basically they are games that have strategies for Alice or Bob.
- Gonshor give another vision that is very connected with Conway's view. It introduce the sign expansion. Length of this sequence correspond to the birthday of the numbers in Conway's view.
- After defining operations (×, +, a → ω^a) we can define expansion with series. Gonshor has made a connection with the sign expansion that is "natural".

Quentin Guilmant

Introduction Numbers

Surreal Numbers

Definitions and operations

Sub-structure and Hahn series

Gaps in the surrea fields.

Computations with surreal numbers.

existing methods

Cauchy completion Computable Analysi

Problem of

Motivation : Analo

Some previous tries

Handle the gaps, new notion of compacity.

Strongly continuou functions

Perspectiv





Surreal numbers Surreal Numbers Definitions and operations Surreal tree

 $[\alpha \mid \alpha]$

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations

Sub-structure and Hahn series

Gaps in the surrea fields.

Computations

numbers, existing

Cauchy completion Computable Analys

over surreal fields

Problem of integration

Motivation : Analog computing

Some previous tries

Handle the gaps, new notion of compacity.

Strongly continuou functions

Perspectiv





Surreal numbers Surreal Numbers Definitions and operations

Voir la figure

Surreal tree

Quentin Guilmant

Introduction Numbers

Surreal Numbers

Definitions and operations

Sub-structure and Hahn series

Gaps in the surrea fields.

Computations with surreal

existing methods

Cauchy completion Computable Analysis

Problem of

Motivation : Analog

computing

Some previous tries

new notion of compacity.

Strongly continuou functions

Perspectiv



Surreal numbers Surreal Numbers Definitions and operations



Quentin Guilmant

Introduction Numbers

Surreal Numbers

Definitions and operations

Sub-structure and Hahn series

Gaps in the surrea fields.

Computations with surreal

numbers, existing

Cauchy completion Computable Analys

over surreal fields

Problem of integration

Motivation : Analog computing

Some previous tries

Handle the gaps, new notion of compacity.

Strongly continuou functions

Perspective



Surreal numbers Surreal Numbers Definitions and operations Surreal tree Surreal tree

쾯

Quentin Guilmant

Introduction Numbers

Surreal Numbers

Definitions and operations

Sub-structure and Hahn series

Gaps in the surreaties fields.

with surreal numbers, existing methods Cauchy completion Computable Analysis

Problem of

Motivation : Analog computing

Some previous tries

Handle the gaps, new notion of compacity.

Strongly continuou functions



Surreal numbers Surreal Numbers Definitions and operations Surreal tree



Quentin Guilmant

Introduction Numbers

Surreal Numbers

Definitions and operations

Sub-structure and Hahn series

Gaps in the surrea fields.

Computations with surreal numbers, existing nethods Cauchy completion Computable Analysis

Problem of integration

Motivation : Analog computing

Some previous tries

Handle the gaps, new notion of compacity.

Strongly continuou functions





Surreal numbers Surreal Numbers Definitions and operations Surreal tree



Quentin Guilmant

Introduction Numbers

Surreal Numbers

Definitions and operations

Sub-structure and Hahn series

Gaps in the surrea fields.

with surreal numbers, existing methods Cauchy completion Computable Analysis

Problem of integration

Motivation : Analog computing

Some previous tries

Handle the gaps, new notion of compacity.

Strongly continuou functions





Surreal numbers Surreal Numbers Definitions and operations Surreal tree



Quentin Guilmant

Definitions and operations

fields.

Motivation : Analog



_

+

÷

_

٠

:

0

+

•

:

 $^{-2}$

_

•

:



_

0

_

÷

+

÷

2

+

•

:

_

÷

Surreal numbers 2019-11-26 Surreal Numbers -Definitions and operations -Surreal tree



Quentin Guilmant

Introduction Numbers

Surreal Numbers

Definitions and operations

Sub-structure and Hahn series

Gaps in the surreal fields.

Computations with surreal numbers, existing nethods Cauchy completion

nputable Analysis r surreal fields _

.

•

 $-\omega$

Problem of integration

Motivation : Analog computing

Some previous trie

Handle the gaps, a new notion of compacity.

Strongly continuous functions



Surreal tree 0 +_ +_ Т $^{-2}$ ___ _ 2 0 0 _ ++_ ++_ ٠ • • . ٠ ٠ ٠ . : • . • : • . . Т ω

ω

 $-\omega$

Surreal numbers Surreal Numbers Definitions and operations



Quentin Guilmant

Introduction Numbers

Surreal Numbers

Definitions and operations

Sub-structure and Hahn series

Gaps in the surrea fields.

computations with surreal numbers, existing nethods Cauchy completion

Computable Analysis over surreal fields

Problem of integration

Motivation : Analog computing

Some previous tries

Handle the gaps, a new notion of compacity.

Strongly continuous functions

Perspective



Surreal tree

Surreal numbers Surreal Numbers Definitions and operations



Quentin Guilmant

Introduction Numbers

Surreal Numbers

Definitions and operations

Sub-structure and Hahn series

Gaps in the surreal fields.

vith surreal numbers, existing nethods Cauchy completion

Computable Analysis over surreal fields

Problem of integration Motivation : Analog computing Some previous tries Handle the gaps, a new notion of compacity.

Strongly continu functions



Surreal numbers Surreal Numbers Definitions and operations



Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations Sub-structure and Hahn series Theorem (Conway)

with surreal numbers, existing methods Cauchy completion Computable Analysis over surreal fields

Problem of integration Motivation : Ana computing

Some previous tries

Handle the gaps, new notion of compacity.

Strongly continuo functions

Perspective

Inductive definitions

If L < R are two subsets of the surreal numbers then there is a

unique x with minimum length such that L < x < R

2019-11-26

Surreal Numbers

Surreal numbers

Inductive definitions

Theorem (Conway) If L < R are two <u>subsets</u> of the surreal numbers then there is a unique x with minimum length such that L < x < R

Ok, now we know how surreal numbers are built but now what about operations? If we speak about numbers, we need addition, multiplication... First we have a theorem by Conway that enable us to build the surreal numbers.

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations

Sub-structure and Hahn series Gaps in the surreal fields

Computations with surreal numbers, existing methods Cauchy completion Computable Analysis

Problem of integration Motivation : Analo computing Some previous trie Handle the gaps, a new notion of compacity

Strongly continu functions

Perspectiv

Inductive definitions

Theorem (Conway)

If L < R are two <u>subsets</u> of the surreal numbers then there is a unique x with minimum length such that L < x < R

• Canonical Conway-representation : $x = [L_x | R_x]$ with $L_x = \{ y \triangleleft x | y < x \}$ and $R_x = \{ y \triangleleft x | y > x \}$ Surreal numbers Surreal Numbers Definitions and operations Surreal Numbers Definitions and operations

Theorem (Conway) $H \perp < R$ are two <u>indicets</u> of the surreal numbers then there is a unique x with minimum length such that L < x < R• Canonical Conway-representation : $x = [L_i \mid R_i]$ with $L_i = \{y < x \mid y < x\}$ and $R_i = \{y < x \mid y > x\}$

Inductive definitions

Ok, now we know how surreal numbers are built but now what about operations? If we speak about numbers, we need addition, multiplication... First we have a theorem by Conway that enable us to build the surreal numbers.

• But the first definition of Conway is very natural it will be the canonical form. But if we want to define operations, it can be difficult to give the canocial representation of the image. That's why the the previous theorem is very very interesting.

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations

Sub-structure and Hahn series Gaps in the surreal

Computations with surreal numbers, existing methods Cauchy completion Computable Analysis

Motivation : Analo computing Some previous trie Handle the gaps, a new notion of compacity.

Strongly continue functions

Inductive definitions

Theorem (Conway)

If L < R are two <u>subsets</u> of the surreal numbers then there is a unique x with minimum length such that L < x < R

- Canonical Conway-representation : $x = [L_x | R_x]$ with $L_x = \{y \triangleleft x | y < x\}$ and $R_x = \{y \triangleleft x | y > x\}$
- Use it to inductively define operations (addition, multiplication, genetic functions...)

Surreal numbers Surreal Numbers Definitions and operations Surreal Numbers Definitions and operations

Theorem (Conway) $K \leq R$ are two <u>subjects</u> of the surreal numbers then there is a unique x with minimum feneth such that $L \leq x \leq R$

Inductive definitions

impue x with minimum length such that L < x < R• Canonical Conveys-representation: $x = [L_x | R_d]$ with $L_x = \{y \lhd x \mid y < x\}$ and $R_x = \{y \lhd x \mid y > x\}$ • Use it to inductively define operations (addition, multiplication, genetic functions...)

Ok, now we know how surreal numbers are built but now what about operations? If we speak about numbers, we need addition, multiplication... First we have a theorem by Conway that enable us to build the surreal numbers.

- But the first definition of Conway is very natural it will be the canonical form. But if we want to define operations, it can be difficult to give the canocial representation of the image. That's why the the previous theorem is very very interesting.
- Then we will take the canonical representations to define operations.

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations

Sub-structure and Hahn series Gaps in the surreal

Computations with surreal numbers, existing methods Cauchy completion Computable Analysis

Problem of integration Motivation : Analog computing Some previous tries Handle the gaps, a new notion of compacity.

functions

Inductive definitions

Theorem (Conway)

If L < R are two <u>subsets</u> of the surreal numbers then there is a unique x with minimum length such that L < x < R

- Canonical Conway-representation : $x = [L_x | R_x]$ with $L_x = \{y \triangleleft x | y < x\}$ and $R_x = \{y \triangleleft x | y > x\}$
- Use it to inductively define operations (addition, multiplication, genetic functions...)
- Uniformity property : when the definition works for any Conway-representations of the arguments

Surreal numbers Surreal Numbers Definitions and operations Surreal Numbers Definitions and operations

Theorem (Convay) $H L \in R$ are two <u>subjects</u> of the survel numbers then there is a unique x with minum length such that L < x < R• Cancical Convay-representation: $x = [L_1, [R_2]$, with $L_2 = (y < x) y < x$.) and $R_2 = (y < x) y > x$.) • Using the two inductively define operations (addition, multiplication, generic functions.) • Uniformity property: when the definition works for any <u>Converse representations</u> of the arguments

Ok, now we know how surreal numbers are built but now what about operations? If we speak about numbers, we need addition, multiplication... First we have a theorem by Conway that enable us to build the surreal numbers.

- But the first definition of Conway is very natural it will be the canonical form. But if we want to define operations, it can be difficult to give the canocial representation of the image. That's why the the previous theorem is very very interesting.
- Then we will take the canonical representations to define operations.
- That is good but if we imagine that we want to define composition it would be fine to be sure that any representation is fine to apply the definition. It is the purpose of the uniformity property.

Inductive definitions

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations

Sub-structure and Hahn series Gaps in the surreal

Computations with surreal numbers, existing methods Cauchy completion Computable Analysis

Problem of integration Motivation : Analog computing Some previous tries Handle the gaps, a new notion of compacity.

functions

Inductive definitions

Theorem (Conway)

If L < R are two <u>subsets</u> of the surreal numbers then there is a unique x with minimum length such that L < x < R

- Canonical Conway-representation : $x = [L_x | R_x]$ with $L_x = \{y \triangleleft x | y < x\}$ and $R_x = \{y \triangleleft x | y > x\}$
- Use it to inductively define operations (addition, multiplication, genetic functions...)
- Uniformity property : when the definition works for any Conway-representations of the arguments

Surreal numbers Surreal Numbers Definitions and operations Surreal Numbers Definitions and operations

Theorem (Convay) $H L \in R$ are two <u>subjects</u> of the survel numbers then there is a unique x with minum length such that L < x < R• Cancical Convay-representation: $x = [L_1, [R_2]$, with $L_2 = (y < x) y < x$.) and $R_2 = (y < x) y > x$.) • Using the two inductively define operations (addition, multiplication, generic functions.) • Uniformity property: when the definition works for any <u>Converse representations</u> of the arguments

Ok, now we know how surreal numbers are built but now what about operations? If we speak about numbers, we need addition, multiplication... First we have a theorem by Conway that enable us to build the surreal numbers.

- But the first definition of Conway is very natural it will be the canonical form. But if we want to define operations, it can be difficult to give the canocial representation of the image. That's why the the previous theorem is very very interesting.
- Then we will take the canonical representations to define operations.
- That is good but if we imagine that we want to define composition it would be fine to be sure that any representation is fine to apply the definition. It is the purpose of the uniformity property.

Inductive definitions

Quentin Guilmant

Introduction : Numbers

Surreal Number

Definitions and operations

Sub-structure and Hahn series Definition

Gaps in the surreal fields.

with surreal numbers, existing methods Cauchy completion

Computable Analysis over surreal fields

Problem of integration

Motivation : Analog computing

Some previous tries

Handle the gaps, new notion of compacity.

Strongly continuou functions

Perspective

Surreal operations : Addition

 $x + y = [L_x + y, x + L_y | R_x + y, x + R_y]$

Surreal numbers Surreal Numbers Surreal Numbers Definitions and operations Surreal operations : Addition

${\sf Definition} \ {\sf de} \ +$

Definition $x + y = [L_x + y, x + L_y | R_x + y, x + R_y]$

Quentin Guilmant

Definitions and operations

fields.

Motivation : Analog

Surreal operations : Addition

 $x + y = [L_x + y, x + L_y | R_x + y, x + R_y]$

• Addition has the uniformity property. • (*No*, +) is a commutative group.



Definition de +

Surreal operations : Addition

Definition $x + y = [L_x + y, x + L_y | R_x + y, x + R_y]$ · Addition has the uniformity property. (No. +) is a commutative group.

Proposition

Definition

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations

Sub-structure and Hahn series

Gaps in the surrea fields.

with surreal numbers, existing methods Cauchy completion Computable Analysis ours surrout field

Problem of integration

Motivation : Analog computing

Some previous tries

Handle the gaps, new notion of compacity.

Strongly continuous functions

Perspectives

Surreal operations : Addition, example

$$\begin{array}{rcl} x & = & \omega + \frac{3}{4} & = & \left[\mathbb{N}, \omega, \omega + \frac{1}{2} \mid \omega + 1 \right] \\ y & = & -\frac{7}{2} & = & \left[-4 \mid 0, -1, -2, -3 \right] \end{array}$$

Surreal numbers Surreal Numbers Definitions and operations Surreal operations : Addition, example Surreal operations : Addition,

 $\begin{array}{rcl} x & = & \omega + \frac{3}{4} & = & \left[\mathbb{N}, \omega, \omega + \frac{1}{2} \mid \omega + 1\right] \\ y & = & -\frac{2}{2} & = & \left[-4 \mid 0, -1, -2, -3\right] \end{array}$

example

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations

Sub-structure and Hahn series

Gaps in the surreal fields.

omputations

Then

with surreal numbers, existing methods

Computable Analysis over surreal fields

Problem of integration

Motivation : Analog computing

Some previous tries

Handle the gaps, new notion of compacity.

Strongly continuou functions

Perspectiv

Surreal operations : Addition, example

$$\begin{array}{rcl} x & = & \omega + \frac{3}{4} & = & \left[\mathbb{N}, \omega, \omega + \frac{1}{2} \; \middle| \; \omega + 1 \right] \\ y & = & -\frac{7}{2} & = & \left[-4 \; \middle| \; 0, -1, -2, -3 \right] \\ \left\{ \begin{array}{rrrr} L_x + y & = & \left\{ -3.5, -2.5, -1.5, -0.5, 0.5, \ldots \right. \\ x + L_y & = & \left\{ \omega - \frac{13}{4} \right\} \\ R_x + y & = & \left\{ \omega - \frac{13}{4} \right\} \\ R_x + y & = & \left\{ \omega - \frac{5}{2} \right\} \\ x + R_y & = & \left\{ \omega + \frac{3}{4}, \omega - \frac{1}{4}, \omega - \frac{5}{4}, \omega - \frac{9}{4} \right\} \end{array}$$

 Surreal operations : Addition,

 $x = \omega + \frac{3}{4} = \left[\mathbb{N}, \omega, \omega + \frac{1}{2} \mid \omega + 1\right]$

 $\gamma = -\frac{7}{2}^4 = [-4 \mid 0, -1, -2, -3]$

 $L_x + y = \{-3.5, -2.5, -1.5, -0.5, 0.5, ...\}$

 $x + L_y = \{\omega - \frac{13}{2}\}$

 $\begin{cases}
R_x + y &= \{\omega - \frac{5}{2}\} \\
x + R_y &= \{\omega + \frac{5}{4}, \omega - \frac{1}{4}, \omega - \frac{5}{4}, \omega - \frac{9}{4}\}
\end{cases}$

example

Quentin Guilmant

Definitions and operations

fields.

Then

Motivation : Analog

Surreal operations : Addition, example

$$\begin{array}{rcl} x & = & \omega + \frac{3}{4} & = & \left[\mathbb{N}, \omega, \omega + \frac{1}{2} \; \middle| \; \omega + 1 \right] \\ y & = & -\frac{7}{2} & = & \left[-4 \; \middle| \; 0, -1, -2, -3 \right] \\ \left\{ \begin{array}{rrrr} L_x + y & = & \left\{ -3.5, -2.5, -1.5, -0.5, 0.5, \ldots \right. \\ x + L_y & = & \left\{ \omega - \frac{13}{4} \right\} \\ R_x + y & = & \left\{ \omega - \frac{13}{4} \right\} \\ R_x + R_y & = & \left\{ \omega + \frac{3}{4}, \omega - \frac{1}{4}, \omega - \frac{5}{4}, \omega - \frac{9}{4} \right\} \\ x + R_y & = & \left\{ \omega - \frac{13}{4} \; \middle| \; \omega - \frac{5}{2} \right] \end{array} \end{array}$$

Surreal numbers 2019-11-26 $x = \omega + \frac{3}{4} = \left[\mathbb{N}, \omega, \omega + \frac{1}{2} \mid \omega + 1\right]$ Surreal Numbers $L_x + y = \{-3.5, -2.5, -1.5, -0.5, 0.5, ...\}$ $x + L_y = \{\omega - \frac{13}{2}\}$ -Definitions and operations $x + R_y = \{\omega + \frac{2}{4}, \omega - \frac{1}{4}, \omega - \frac{5}{4}, \omega - \frac{9}{4}\}$ -Surreal operations : Addition, example

Surreal operations : Addition,

 $\gamma = -\frac{7}{2}^4 = [-4 \mid 0, -1, -2, -3]$

 $x + y = \left[\omega - \frac{13}{4} \mid \omega - \frac{5}{2}\right]$

 $R_x + y = {\omega - \frac{2}{5}}$

example

Quentin Guilmant

Definitions and operations

fields.

Then

Motivation : Analog

Surreal operations : Addition, example

$$\begin{array}{rcl} x &=& \omega + \frac{3}{4} &=& \left[\mathbb{N}, \omega, \omega + \frac{1}{2} \mid \omega + 1 \right] \\ y &=& -\frac{7}{2} &=& \left[-4 \mid 0, -1, -2, -3 \right] \\ \left\{ \begin{array}{rrr} L_x + y &=& \left\{ -3.5, -2.5, -1.5, -0.5, 0.5, \ldots \right\} \\ x + L_y &=& \left\{ \omega - \frac{13}{4} \right\} \\ R_x + y &=& \left\{ \omega - \frac{5}{2} \right\} \\ x + R_y &=& \left\{ \omega + \frac{3}{4}, \omega - \frac{1}{4}, \omega - \frac{5}{4}, \omega - \frac{9}{4} \right\} \\ x + y &=& \left[\omega - \frac{13}{4} \mid \omega - \frac{5}{2} \right] \\ x + y &=& \left[(+)^{\omega} - - - + + \mid (+)^{\omega} - - - + \right] \end{array}$$

Surreal numbers 2019-11-26 $x = \omega + \frac{3}{4} = \left[\mathbb{N}, \omega, \omega + \frac{1}{2} \mid \omega + 1\right]$ Surreal Numbers $\gamma = -\frac{7}{2}^4 = [-4 \mid 0, -1, -2, -3]$ $L_x + y = \{-3.5, -2.5, -1.5, -0.5, 0.5, ...\}$ $x + L_y = \{\omega - \frac{13}{2}\}$ -Definitions and operations $R_x + y = {\omega$ $x + R_y = \{\omega + \frac{5}{4}, \omega - \frac{1}{4}, \omega - \frac{5}{4}, \omega - \frac{9}{4}\}$ -Surreal operations : Addition, example $x + y = [(+)^{\omega} - - - + + | (+)^{\omega} - - + +]$

Surreal operations : Addition,

 $x + y = \left[\omega - \frac{13}{4} \mid \omega - \frac{5}{2}\right]$

example

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations

Sub-structure and Hahn series

Gaps in the surreal fields.

with surreal numbers, existing methods Cauchy completion Computable Analysis over surreal fields

Problem of integration Motivation : Analog computing Some previous tries Handle the gaps, a new notion of compacity. Strongly continuous

Perspectiv

Surreal operations : Addition, example

$$\begin{array}{rcl} x &=& \omega + \frac{3}{4} &=& \left[\mathbb{N}, \omega, \omega + \frac{1}{2} \mid \omega + 1 \right] \\ y &=& -\frac{7}{2} &=& \left[-4 \mid 0, -1, -2, -3 \right] \\ \\ \text{Then} &\begin{cases} L_x + y &=& \left\{ -3.5, -2.5, -1.5, -0.5, 0.5, \ldots \right\} \\ x + L_y &=& \left\{ \omega - \frac{13}{4} \right\} \\ R_x + y &=& \left\{ \omega - \frac{5}{2} \right\} \\ x + R_y &=& \left\{ \omega + \frac{3}{4}, \omega - \frac{1}{4}, \omega - \frac{5}{4}, \omega - \frac{9}{4} \right\} \\ & & x + y = \left[\omega - \frac{13}{4} \mid \omega - \frac{5}{2} \right] \\ x + y = \left[(+)^{\omega} - - - + + \mid (+)^{\omega} - - - + \right] \\ \\ \text{Basically we have to choose between} \end{array}$$

 $(+)^{\omega}---++$ and $(+)^{\omega}---+-$

Surreal numbers Surreal Numbers Surreal Numbers Definitions and operations Surreal operations : Addition, example Surreal operations : Addition

 $x = \omega + \frac{3}{2} = \left[\mathbb{N}, \omega, \omega + \frac{1}{2} \mid \omega + 1\right]$

 $\gamma = -\frac{7}{2} = [-4 \mid 0, -1, -2, -3]$

 $x + R_y = \{\omega + \frac{5}{4}, \omega - \frac{1}{4}, \omega - \frac{5}{4}, \omega - \frac{9}{4}\}$

 $\begin{aligned} x + y &= \left[\omega - \frac{13}{4} \right] \omega - \frac{5}{2} \\ x + y &= \left[(+)^{\omega} - - - + + \right] (+)^{\omega} - - - +] \\ \text{Basically we have to choose between} \\ (+)^{\omega} - - - + + + & \text{and} & (+)^{\omega} - - - + - \end{aligned}$

 $x + L_y = \{\omega - \frac{13}{4}\}$

 $R_{-} + \dot{y} = i\omega -$

 $L_v + v = \{-3.5, -2.5, -1.5, -0.5, 0.5, ...\}$

example

Quentin Guilmant

Definitions and operations

fields.

Then

Motivation : Analog

Surreal operations : Addition, example

$$\begin{array}{rcl} x &=& \omega + \frac{3}{4} &=& \left[\mathbb{N}, \omega, \omega + \frac{1}{2} \mid \omega + 1 \right] \\ y &=& -\frac{7}{2} &=& \left[-4 \mid 0, -1, -2, -3 \right] \\ \\ \text{Then} &\begin{cases} L_x + y &=& \{ -3.5, -2.5, -1.5, -0.5, 0.5, \ldots \} \\ x + L_y &=& \{ \omega - \frac{13}{4} \} \\ R_x + y &=& \{ \omega - \frac{5}{2} \} \\ x + R_y &=& \{ \omega + \frac{3}{4}, \omega - \frac{1}{4}, \omega - \frac{5}{4}, \omega - \frac{9}{4} \} \\ & & x + y = \left[\omega - \frac{13}{4} \mid \omega - \frac{5}{2} \right] \\ x + y = \left[(+)^{\omega} - - - + + \mid (+)^{\omega} - - - + \right] \\ \\ \text{Basically we have to choose between} \end{array}$$

 $(+)^{\omega} - - - + + +$ and $(+)^{\omega} - - + -$ Simplicity property :

$$x + y = (+)^{\omega} - - - + - = \omega - \frac{11}{4}$$

Surreal numbers 11-26 Surreal Numbers $L_v + v = \{-3.5, -2.5, -1.5, -0.5, 0.5, ...\}$ -Definitions and operations 2019-1 -Surreal operations : Addition, example $x + y = [(+)^{\omega} - - - + + | (+)^{\omega} - - - +]$ Basically we have to choose between $(+)^{\omega} - - - + + + and (+)^{\omega} - - - + -$ Simplicity property

Surreal operations : Addition

 $x = \omega + \frac{3}{2} = \left[\mathbb{N}, \omega, \omega + \frac{1}{2} \mid \omega + 1\right]$

 $\gamma = -\frac{7}{2} = [-4 \mid 0, -1, -2, -3]$

 $x + R_y = \{\omega + \frac{5}{4}, \omega - \frac{1}{4}, \omega - \frac{5}{4}, \omega - \frac{9}{4}\}$

 $x + y = \left[\omega - \frac{13}{4} \mid \omega - \frac{5}{2} \right]$

 $x + y = (+)^{\omega} - - - + - = \omega - \frac{11}{4}$

 $x + L_y = \{\omega - \frac{13}{4}\}$

 $R_{-} + \dot{y} = i\omega -$

example

Quentin Guilmant

fields.

Definitions and operations

Definition

 $x \times y = \begin{bmatrix} l_x y + xl_y - l_x l_y \\ r_x y + xr_y - r_x r_y \end{bmatrix} \begin{bmatrix} l_x y + xr_y - l_x r_y \\ r_x y + xl_y - r_x l_y \end{bmatrix}$

Surreal numbers 2019-11-26 Surreal Numbers -Definitions and operations -Surreal operations : Multiplication

Definition de \times

Surreal operations : Multiplication

 $x \times y = \begin{bmatrix} l_x y + xl_y - l_x l_y \\ r_x y + xr_y - r_x r_y \end{bmatrix} \begin{bmatrix} l_x y + xr_y - l_x r_y \\ r_x y + xr_y - r_x r_y \end{bmatrix}$

Surreal operations : Multiplication

Motivation : Analog

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

fields.

operations and Definition

Sub-structure and Hahn series Gaps in the surreal

Motivation : Analog

 $x \times y = \begin{bmatrix} l_x y + xl_y - l_x l_y \\ r_x y + xr_y - r_x r_y \end{bmatrix} \begin{bmatrix} l_x y + xr_y - l_x r_y \\ r_x y + xl_y - r_x l_y \end{bmatrix}$

Proposition

• Multiplication has the uniformity property

Surreal operations : Multiplication

• ($No_{<,}+,\times$) is field.

Surreal operations : Multiplication

 $x \times y = \begin{bmatrix} l_x y + xl_y - l_x l_y \\ r_x y + xr_y - r_x r_y \end{bmatrix} \begin{bmatrix} l_x y + xr_y - l_x r_y \\ r_x y + xr_y - r_x r_y \end{bmatrix}$

Multiplication has the uniformity property
 (No₅, +, ×) is field.

Definition de \times
Quentin Guilmant

Introduction : Numbers

Proposition

• $\langle \rangle = 0$

defined as follows.

Surreal

Definitions and operations

Sub-structure and Hahn series Gaps in the surreal

Computations with surreal numbers, existing nethods Cauchy completion Computable Analysis over surreal fields

Problem of ntegration Motivation : Analog computing Some previous tries Handle the gaps, a new notion of compacity. Strongly continuous functions

Surreal operations : Inverse

Let $x = [L_x \mid R_x]$ in the canonical representation. Then $\frac{1}{x}$ is

 $\langle y_0,\ldots,y_n\rangle=\frac{1-(x-y_n)\langle y_0,\ldots,y_{n-1}\rangle}{y_n}$

 $\begin{cases} L_{1/x} = \{ \langle y_0, \dots, y_n \rangle \mid | \{i \mid y_i \in L_x\} \mid \in 2\mathbb{N} \} \\ R_{1/x} = \{ \langle y_0, \dots, y_n \rangle \mid | \{i \mid y_i \in L_x\} \mid \in 2\mathbb{N} + 1 \} \end{cases}$



Definition de l'inverse

Surreal operations : Inverse

And the definition has the uniformity property.

Then $\frac{1}{x} = \begin{bmatrix} L_{1/x} & R_{1/x} \end{bmatrix}$

• For $y_0, \ldots, y_n \in (L_x \cup R_x) \setminus \{0\}$,

Quentin Guilmant

ntroduction Numbers

Surreal

Definitions and operations

Sub-structure and Hahn series Gaps in the surreal

Computations with surreal numbers, existing methods Cauchy completion Computable Analysis over surreal fields

Problem of integration Motivation : Ana computing

Some previous tries

Handle the gaps, new notion of compacity.

Strongly continuou functions

Perspective

Exponential function

Theorem (Gonshor [6])

It is possible to extend the function $\exp : \mathbb{R} \to \mathbb{R}$ to exp : $No \to No$ such that it has an inductive definition with the uniformity property. Surreal numbers Surreal Numbers Definitions and operations Exponential function Exponential function Theorem (Gonshor [6]) It is possible to extend the function $\exp : \mathbb{R} \to \mathbb{R}$ to

It is possible to extend the function exp : $\mathbb{R} \to \mathbb{R}$ to exp : $\mathbf{No} \to \mathbf{No}$ such that it has an inductive definition with the uniformity property.

• Even the exponential function has a nice definition ! In particular it is inductive with uniform property.

Quentin Guilmant

ntroduction Numbers

Surreal

Definitions and operations

Sub-structure and Hahn series Gans in the surreal

Computations with surreal numbers, existing nethods Cauchy completion Computable Analysis over surreal fields

Theorem (Gonshor [6])

It is possible to extend the function $\exp : \mathbb{R} \to \mathbb{R}$ to exp : $No \to No$ such that it has an inductive definition with the uniformity property.

Exponential function

Proposition (Van den Dries, Ehrlich [8])

There is a hierarchy of elementary extensions $\mathbb{R} \subseteq \mathbf{No}_{<\lambda} \subseteq \mathbf{No}$ (for λ an ε -number) for the language of ordered fields together with exp (and restricted analytic functions).

Surreal numbers Surreal Numbers Definitions and operations Exponential function

Exponential function

Theorem (Gonshor [6])

It is possible to extend the function $\exp : \mathbb{R} \to \mathbb{R}$ to $\exp : \mathbf{No} \to \mathbf{No}$ such that it has an inductive definition with the uniformity property.

Proposition (Van den Dries, Ehrlich [8])

There is a hierarchy of elementary extensions $\mathbb{R} \subseteq \mathbf{No}_{c,\lambda} \subseteq \mathbf{No}$ (for λ an ε -number) for the language of ordered fields together with exp (and restricted analytic functions).

- Even the exponential function has a nice definition ! In particular it is inductive with uniform property.
- The definition of the exponential function is very satisfying, it has all the suitable first order properties of the exponential function.

Quentin Guilmant

ntroduction Numbers

Surreal

Definitions and operations

Sub-structure and Hahn series Gaps in the surreal

Computations with surreal numbers, xisting nethods Cauchy completion Computable Analysis over surreal fields

Problem of ntegration Motivation : Analog computing Some previous tries Handle the gaps, a new notion of compacity. Strongly continuous

Exponential function

Theorem (Gonshor [6])

It is possible to extend the function $\exp : \mathbb{R} \to \mathbb{R}$ to exp : $No \to No$ such that it has an inductive definition with the uniformity property.

Proposition (Van den Dries, Ehrlich [8])

There is a hierarchy of elementary extensions $\mathbb{R} \subseteq \mathbf{No}_{<\lambda} \subseteq \mathbf{No}$ (for λ an ε -number) for the language of ordered fields together with exp (and restricted analytic functions).

Warning

sin and cos do not admit extension to surreal number. You would need to give sense to $\exp(i\omega)$. Surreal numbers Surreal Numbers Definitions and operations Exponential function

Theorem (Gonshor [6]) It is possible to extend the function exp: $\mathbb{R} \to \mathbb{R}$ to exp: $No \to No$ such that it has an inductive definition with the uniformity property.

Exponential function

Proposition (Van den Dries, Ehrlich [8])

There is a hierarchy of elementary extensions $\mathbb{R} \subseteq \mathbf{No}_{c,\lambda} \subseteq \mathbf{No}$ (for λ an ε -number) for the language of ordered fields together with exp (and restricted analytic functions).

sin and cos do not admit extension to surreal number You would need to give sense to $exp(i\omega)$.

- Even the exponential function has a nice definition ! In particular it is inductive with uniform property.
- The definition of the exponential function is very satisfying, it has all the suitable first order properties of the exponential function.
- What does it mean to orbit ω times around the origin ? Even worth : $\sqrt{\omega}$ times ?

Quentin Guilmant

Introduction Numbers

Surreal Numbers Notation

Definitions and operations

Sub-structure and Hahn series

Gaps in the surrea fields.

with surreal numbers,

existing methods Cauchy completi

Computable Analysis over surreal fields

Problem of integration Motivation : Ana computing

Some previous tries

Handle the gaps, new notion of compacity.

Strongly continuou functions

Perspective

Surreal numbers of bounded length

 $No_{<\alpha} = \{x \in No \mid length(x) < \alpha\}$

Surreal numbers Surreal Numbers Surreal Numbers Sub-structure and Hahn series Sub-structure and Hahn series Surreal numbers of bounded length Surreal numbers of bounded length

ion $No_{<\alpha} = \{x \in No \mid length(x) < \alpha\}$

- So far we were interested in the whole class of surreal numbers. But there are sub-structures that may be interesteing. Typically, what about the set of surreal numbers of bounded length (or birthday).
- The very first question is, what are the conditions on α so that we have usual algebraic structures.

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations

Sub-structure and Hahn series

Gaps in the surreal fields.

```
Computations
vith surreal
numbers,
xisting
nethods
Cauchy completion
Computable Analysis
```

Computable Analysis over surreal fields Problem of

Motivation : Analo computing

Some previous tries Handle the gaps, a new notion of compacity.

functions

Surreal numbers of bounded length

 $No_{<\alpha} = \{x \in No \mid length(x) < \alpha\}$

Proposition (Van den Dries and Ehrlich [8], corollaries 3.1, 4.4 and 4.9)

No $_{<\lambda}$ is

Notation

- an additive group iff λ is additive (i.e has form λ = ω^α for some ordinal α)
- is a ring iff λ is multiplicative (i.e has form $\lambda = \omega^{\omega^{\alpha}}$ for some ordinal α)
- is a field iff λ is an ε -number (i.e satisfies $\lambda = \omega^{\lambda}$)

Surreal numbers Surreal Numbers Sub-structure and Hahn series Sub-structure and Hahn series Surreal numbers of bounded length

Notation
$$\begin{split} & \text{No}_{n,n} = \{x \in \text{No} \mid \text{keepth}(x) < \alpha\} \\ & \text{Proposition} (Van den Dries and Ehrlich [8], corollaries 3.1, \\ 4 & and 4.3) \\ & \text{No}_{n,k} \\ & an additive group iff \lambda is additive (i.e. has form <math>\lambda = \omega^{n}$$
 for one regardly λ is multiplicative (i.e. has form $\lambda = \omega^{n^{n}}$ for some gradies).

is a field iff λ is an ε-number (i.e. satisfies λ = ω^λ)

Surreal numbers of bounded length

- So far we were interested in the whole class of surreal numbers. But there are sub-structures that may be interesteing. Typically, what about the set of surreal numbers of bounded length (or birthday).
- The very first question is, what are the conditions on α so that we have usual algebraic structures.
- The answer is very intuitive : it is basically the desired property applied to the ordinal that bounds the length.

Quentin Guilmant

Introduction Numbers

Surreal Numbers

Definitions and operations

Sub-structure and Hahn series

Gaps in the surreal fields.

with surreal numbers, existing methods Cauchy completion

Computable Analysis over surreal fields

Problem of integration Motivation : Ana

computing

Handle the gaps, a

compacity.

functions

Perspecti

Interlude : Hahn series

Definition (Hahn series field)

Let \mathbb{K} be a field and Γ an Abelian ordered group. The associated formal power series field denoted $\mathbb{K}((t^{\Gamma}))$ is

 $| r_{\gamma} \in \mathbb{K}, \text{ supp}(x) := \{ \gamma | r_{\gamma} \neq 0 \} \text{ is well ordered} \}$ $\left\{\sum_{\gamma \in \Gamma} r_{\gamma} t^{\gamma}\right\}$

Surreal numbers 11-26 Surreal Numbers -Sub-structure and Hahn series 2019-1 └─Interlude : Hahn series

Interlude : Hahn series

$\begin{array}{l} \mbox{Definition (Hahn series field)}\\ \mbox{Let K be a field and Γ an Abelian ordered group. The associated formal power series field denoted $K(t^T)$ is $\left\{\sum\limits_{y \in T} r_y t^Y \mid z_y \in \mathbb{K}, \mbox{supp}(x) := \{\gamma \mid z_y \neq 0\} $ is well ordered $R_1(t^T)$ ordered $R_2(t^T)$ is $t = 1 \ t = 1 \$

So far we have seen the two first definition of the surreal numbers. What about the third one?

• We first define Hahn series.

Quentin Guilmant

Introduction Numbers

Surreal Numbers

Definitions and operations

Sub-structure and Hahn series

Gaps in the surreal fields.

vith surreal umbers, xisting nethods Cauchy completion Computable Analysis

Problem of integration Motivation : Anal computing Some previous trie Handle the gaps

Handle the gaps, new notion of compacity.

Strongly contin functions

Perspecti

Interlude : Hahn series

Definition (Hahn series field)

Let \mathbb{K} be a field and Γ an Abelian ordered group. The associated formal power series field denoted $\mathbb{K}((t^{\Gamma}))$ is

 $\left\{\sum_{\gamma\in\Gamma}r_{\gamma}t^{\gamma}\mid r_{\gamma}\in\mathbb{K}, \text{ supp}(x):=\{\gamma\mid r_{\gamma}\neq 0\} \text{ is well ordered}\right\}$ This fields admits a natural notion of order, the lexicographical order. Surreal numbers Surreal Numbers Surreal Numbers Sub-structure and Hahn series

Definition (Hahn series field) Let K be a field and Γ as Abbilan ordered group. The associated formal power series field denoted $\mathbb{K}([t^7])$ is $\begin{cases} \sum_{i=1}^{r} r_i t^2 \\ \sum_{i=1}^{r} r_i t^2 \end{cases}$, $x_i \in \mathbb{K}$, $x_{ipp}(x_i) := \{\gamma \mid x_i \neq 0\}$ is well ordered This fields admits a natural notion of order, the lesiconrabic:

Interlude : Hahn series

So far we have seen the two first definition of the surreal numbers. What about the third one?

- We first define Hahn series.
- This is an ordered field

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations

Sub-structure and Hahn series

Computations with surreal numbers, wisting nethods Cauchy completion Computable Analysis over surreal fields

Problem of ntegration Motivation : Analog computing Some previous tries Handle the gaps, a new notion of compacity. Strongly continuous

Interlude : Hahn series

Definition (Hahn series field)

Let \mathbb{K} be a field and Γ an Abelian ordered group. The associated formal power series field denoted $\mathbb{K}((t^{\Gamma}))$ is

 $\left\{\sum_{\gamma\in\Gamma}r_{\gamma}t^{\gamma}\mid r_{\gamma}\in\mathbb{K}, \text{ supp}(x):=\{\gamma\mid r_{\gamma}\neq 0\} \text{ is well ordered}\right\}$ This fields admits a natural notion of order, the lexicographical order.

Theorem (Gonshor, [6])

Every x in **No** can be written in a unique way as $x = \sum_{i < \alpha} r_i \omega^{a_i}$ with $r_i \in \mathbb{R}$ and the $a_i \in \mathbf{No}$ decreasing and $\mathbf{No} \simeq \mathbb{R}((t^{\mathbf{No}}))$ (ordered fields isomorphism). Surreal numbers Surreal Numbers Sub-structure and Hahn series Interlude : Hahn series

Definition (Hahn series field) Let K be a field and T an Abdian ordered group. The activated from a power views field denoted $K((r^3))$ is $\left\{\sum_{ij} r_i r^{ij} \mid i \in \mathbb{K}, useq(z) := (\gamma \mid r_i \neq 0)$ is well ordered This field advices a natural action of order, the isocographical order. Those (Constor, 16)

Every x in **No** can be written in a unique way as $x = \sum_{i < 0} r_i \omega^i$ with $r_i \in \mathbb{R}$ and the $a_i \in No$ decreasing and $No \simeq \mathbb{R}((t^{No}))$ (ordered fields isomorphism).

So far we have seen the two first definition of the surreal numbers. What about the third one $? \end{tabular}$

- We first define Hahn series.
- This is an ordered field
- We have a very good isomorphism that links the sign expansions and the Hahn series. Moreover the the sign expansion can be "easily" deduced from the Hahn series. Finally the Hahn series of ordinal number seen as surreal numbers are their normal forms as usual ordinals.

Interlude : Hahn series

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations

Sub-structure and Hahn series

Gaps in the surreal ields.

```
with surreal
numbers,
existing
methods
Cauchy completion
Computable Analysis
```

Problem of integration Motivation : Analcomputing Some previous trie

Handle the gaps, new notion of compacity.

Strongly continue functions

Perspectiv

Sub-fields from Hahn series

Theorem (Alling [1], Van den Dries, Ehrlich [8])

Let $\mathbb K$ be real-closed and Γ divisible. Let λ be an $\varepsilon\text{-number}.$ Then

 $\mathbb{K}_{\lambda}^{\Gamma} := \left\{ x \in \mathbb{K}((t^{\Gamma})) \mid \text{supp } x \text{ has order type lower than } \lambda \right\}$ is a real-closed field.

Surreal numbers Surreal Numbers Sub-structure and Hahn series Sub-structure from Hahn series Sub-fields from Hahn series

 $\begin{array}{l} \label{eq:constraints} \hline \textbf{Theorem} \mbox{ (Alling [1], Van den Dries, Ehrlich [8])} \\ Let <math display="inline">\mathbb{K} \mbox{ be real-closed and } \Gamma \mbox{ divisible. Let } \lambda \mbox{ be an } \varepsilon\mbox{-number.} \\ \hline \textbf{Then} \\ \mathbb{K}^{\Gamma}_{\lambda} := \left\{ x \in \mathbb{K}(\{r^{0}\}) \ \middle| \mbox{ supp } x \mbox{ has order type lower than } \lambda \right\} \\ \text{is a real-closed field.} \end{array}$

- Alling worked with cardinal numbers as length for the sums but with $\mathbb K$ a real-closed field, Ehrlich and Van den Dries with $\mathbb R$ but with ordinal length.
- This second theorem enable us to make a link between the two types of field we have seen so far.

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations

Sub-structure and Hahn series

Gaps in the surrea fields.

Computations with surreal numbers, existing methods Cauchy completion Computable Analysis over surreal fields

Problem of integration Motivation : Analog computing Some previous tries Handle the gaps, a new notion of compacity. Strongly continuous functions

Sub-fields from Hahn series

Theorem (Alling [1], Van den Dries, Ehrlich [8])

Let $\mathbb K$ be real-closed and Γ divisible. Let λ be an $\varepsilon\text{-number}.$ Then

 $\mathbb{K}_{\lambda}^{\Gamma} := \left\{ x \in \mathbb{K}((t^{\Gamma})) \mid \text{supp } x \text{ has order type lower than } \lambda \right\}$ is a real-closed field.

Theorem (Van den Dries, Ehrlich [8])

If λ is an ε -number then

 $oldsymbol{No}_{<\lambda}\simeqigcup_{\mu<\lambda}\mathbb{R}_\lambda^{oldsymbol{No}_{<\mu}}$

where μ ranges over multiplicative ordinals. If λ is a regular cardinal then $\pmb{No}_{<\lambda}\simeq\mathbb{R}_{\lambda}^{\pmb{No}_{<\lambda}}$

Surreal numbers Surreal Numbers Sub-structure and Hahn series Sub-fields from Hahn series

Theorem (Alling [1], Van den Dries, Ehrlich [8]) Let K be raid-cload and Γ divisible. Let λ be an (-number, Than $K_{\lambda}^{c} := \{x \in \mathbb{K}(\{\Gamma\})\}$ supp shail order type lower than λ } is a raid-cload field. Theorem (Van den Dries, Ehrlich [8]) $\pi' \lambda$ is an (-number than π

Sub-fields from Hahn series

 $\mathbf{No}_{c,\lambda} \simeq \bigcup_{\mu \in \mathbb{Z}} \mathbb{R}_{\lambda}^{\mathbf{No}_{c,\mu}}$ where μ ranges over multiplicative ordinals. If λ is a regula cardinal then $\mathbf{No}_{c,\lambda} \simeq \mathbb{R}_{\lambda}^{\mathbf{No}_{c,\lambda}}$

- Alling worked with cardinal numbers as length for the sums but with $\mathbb K$ a real-closed field, Ehrlich and Van den Dries with $\mathbb R$ but with ordinal length.
- This second theorem enable us to make a link between the two types of field we have seen so far.

Quentin Guilmant

Introduction : Numbers

Surreal Numbers Definition

Definitions and operations

Sub-structure an Hahn series

Gaps in the surreal fields.

with surreal numbers, existing methods

Cauchy completion Computable Analysis over surreal fields

Problem of integration Motivation : Analog

computing

Some previous tries

new notion of compacity.

Strongly continuou functions

Perspective

Cuts and gaps

• A <u>cut</u> is a couple of sets $L, R \subseteq \mathbb{K} \subseteq \mathbf{No}$ such that L < R.

Surreal numbers Surreal Numbers Gaps in the surreal fields. Cuts and gaps

Définition cut, gap et Cauchy-gap

Cuts and gaps

• A <u>cut</u> is a couple of sets $L, R \subseteq K \subseteq No$ such that L < R.

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definition

Definitions and operations

Sub-structure and Hahn series

Gaps in the surreal fields.

with surreal numbers, existing methods Cauchy completion Computable Analysis over surreal fields

Problem of integration Motivation : Analog

computing

Handle the gaps, new notion of compacity

Strongly continuou functions

Perspectiv

Cuts and gaps

A <u>cut</u> is a couple of sets L, R ⊆ K ⊆ No such that L < R.
A cut L < R is a gap of K if [L | R] ∉ K. It is non-trivial

if L has no maximum and R has no minimum.

Surreal numbers Surreal Numbers Gaps in the surreal fields.

Définition cut, gap et Cauchy-gap

A <u>cast</u> is a couple of sets L, R ⊆ K ⊆ No such that L < R.
 A cut L < R is a <u>gap</u> of K if [L | R] ∉ K. It is non-trivial if L has no maximum and R has no minimum.

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definition

Definitions and operations

Sub-structure and Hahn series

Gaps in the surreal fields.

omputations /ith surreal umbers, xisting nethods

Computable Analysis over surreal fields

Problem of integration Motivation : Analo computing Some previous trie Handle the gaps, a new potion of

compacity. Strongly continuo

Perspectiv

Cuts and gaps

Surreal numbers Surreal Numbers Gaps in the surreal fields. Cuts and gaps

Définition cut, gap et Cauchy-gap

Cuts and gaps

A <u>cut</u> is a couple of sets L, R ⊆ K ⊆ No such that L < R.
 A cut L < R is a <u>gap</u> of K if [L | R] ∉ K. It is non-trivial if L has no mizeimum and R has no minimum.

Example

If $L = \mathbb{N}$ and $R = \left\{ \omega^a \mid a \in (\mathbf{No}_{<\mu})^+_+ \right\}$, then L < R is a gap of $\mathbb{R}^{\mathbf{No}_{<\mu}}_{\lambda}$. This special gap is denoted ∞ .

Example If $L = \mathbb{N}$ and $R = \left\{ \omega^a \mid a \in (\mathbf{No}_{<\mu})^*_+ \right\}$, then L < R is a gap of $\mathbb{R}^{\mathbf{No}_{<\mu}}_{\lambda}$. This special gap is denoted ∞ .

if L has no maximum and R has no minimum.

A <u>cut</u> is a couple of sets L, R ⊆ K ⊆ No such that L < R.
A cut L < R is a gap of K if [L | R] ∉ K. It is non-trivial

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations

- Sub-structure and Hahn series
- Gaps in the surreal fields.

Computations with surreal numbers, existing nethods Cauchy completion Computable Analysis over surreal fields

Problem of ntegration Motivation : Analog computing Some previous tries Handle the gaps, a new notion of compacity. Strongly continuous

Perspectives

Cuts and gaps



Définition cut, gap et Cauchy-gap

Cuts and gaps

ion

A <u>cut</u> is a couple of sets L, R ⊆ K ⊆ No such that L < R.
 A cut L < R is a <u>gap</u> of K if [L | R] ∉ K. It is non-trivial if L has no maximum and R has no minimum.

Example

If $L = \mathbb{N}$ and $R = \{\omega^a \mid a \in (\mathbf{No}_{<\mu})^*_+\}$, then L < R is a gap of $\mathbb{R}_1^{\mathbf{No}_{<\mu}}$. This special gap is denoted ∞ .

Definition (Cauchy Cut)

L < R is a Cauchy-cut of K if for all $\varepsilon \in \mathbb{K}^*_+$ there are $l \in L$ and $r \in R$ such that $r - l < \varepsilon$.

- A <u>cut</u> is a couple of sets $L, R \subseteq \mathbb{K} \subseteq \mathbf{No}$ such that L < R.
- A cut L < R is a gap of K if [L | R] ∉ K. It is non-trivial if L has no maximum and R has no minimum.

Example

Definition

If $L = \mathbb{N}$ and $R = \left\{ \omega^a \mid a \in (\mathbf{No}_{<\mu})^*_+ \right\}$, then L < R is a gap of $\mathbb{R}^{\mathbf{No}_{<\mu}}_{\lambda}$. This special gap is denoted ∞ .

Definition (Cauchy Cut)

L < R is a Cauchy-cut of \mathbb{K} if for all $\varepsilon \in \mathbb{K}_+^*$ there are $l \in L$ and $r \in R$ such that $r - l < \varepsilon$.

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

operations Sub-structure and

Gaps in the surreal fields. Proposition (Conway, [3])

Gaps in **No** may be of the form :

For $x \in \mathbf{No}$ and \mathcal{G} a gap $x + \omega^{\mathcal{G}}$

Computations with surreal numbers, existing methods Cauchy completion Computable Analysis over surreal fields

Problem of integration

Motivation : Anal computing

Some previous tries

new notion of compacity.

Strongly continuo functions

Perspectiv

Types of Gaps in $\ensuremath{\textbf{No}}$

 $\sum_{i \in Ord} r_i \omega^{a_i}$

Surreal numbers Surreal Numbers Gaps in the surreal fields. Gaps of Gaps in **No**

(Type 1)

(Type 2)

Proposition (Conway, [3]) Gaps in No may be of the form : $\sum_{k \in Ord} r_{kk} a^{k}$

Types of Gaps in No

Les deux types de gap

Please notice that there are only non trivial gaps. That is because if there were a non-trivial gap, one of the set L or R involved in the gap would actually be a proper class.

Quentin Guilmant

fields.

3 Computations with surreal numbers, existing methods

Computations with surreal numbers, existing methods

Motivation : Analog

Outline

Surreal numbers 2019-11-26 Computations with surreal numbers, existing me-Occupations with surreal numbers, existing methods thods -Outline

Outline

Quentin Guilmant

Introduction Numbers

Surreal Numbers

Definitions and operations Sub-structure and Habn series

Gaps in the surrea fields.

Computations with surreal numbers, existing methods

Cauchy completion

Computable Analysis over surreal fields

Problem of integration Motivation : Ana computing

Some previous tries

Handle the gaps, new notion of compacity.

Strongly continuo functions

Perspecti

Cauchy cuts and Cauchy completion

K is Cauchy-complete *iff* it has no non-trivial Cauchy gap. $\mathbb{R}^{\mu}_{\lambda} := \mathbb{R}^{No<\mu}_{\lambda}$ has a simple Cauchy-completion. Surreal numbers Computations with surreal numbers, existing methods

Cauchy completion

-26

2019-11

Cauchy cuts and Cauchy completion

Cauchy cuts and Cauchy completion

 $\mathbb K$ is Cauchy-complete $i\!\! f\!\! f$ it has no non-trivial Cauchy gap $\mathbb R^\mu_\lambda:=\mathbb R^{Na+\mu}_\lambda$ has a simple Cauchy-completion.

• If we speak about Cauchy gap, it is because we can avoid them. In particular we can have a complete field. The difference with a real case being that the sequence must have length the degree of the field, which is the coinitiality of the set of positive elements of the field.

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations Sub-structure and Hahn series Gaps in the surreal fields.

Computations with surreal numbers, existing methods

Cauchy completion

Computable Analysis over surreal fields

Problem of integration Motivation : Anal computing Some previous trie

Handle the gaps, new notion of compacity.

Strongly continuou functions

Perspective

Cauchy cuts and Cauchy completion

 \mathbb{K} is Cauchy-complete *iff* it has no non-trivial Cauchy gap. $\mathbb{R}^{\mu}_{\lambda} := \mathbb{R}^{No<\mu}_{\lambda}$ has a simple Cauchy-completion.

Proposition

The Cauchy-completion of $\mathbb{R}^{\mu}_{\lambda}$ is $\widetilde{\mathbb{R}^{\mu}_{\lambda}} = \mathbb{R}^{\mu}_{\lambda} \cup \left\{ \sum_{i < \lambda} r_{i} \omega^{a_{i}} \mid r_{i} \in \mathbb{R}, a_{i} \in \mathbf{No}_{<\mu} \text{ are coinitial} \right\}$ Surreal numbers Computations with surreal numbers, existing methods Cauchy completion

-Cauchy cuts and Cauchy completion

-26

2019-11

$\begin{array}{c} \mbox{Cauchy cuts and Cauchy} \\ \mbox{completion} \\ \mbox{K is Cauchy-complete iff it has no non-trivial Cauchy gap.} \\ \mbox{R}^{Macro}_{\mu} := \mbox{R}^{Macro}_{\mu} \mbox{ has a simple Cauchy-completion.} \end{array}$

 $\mathbb{R}^{d}_{\lambda} := \mathbb{R}^{dev}_{\lambda}$ has a simple Cauchy-completion. Proposition The Cauchy-completion of \mathbb{R}^{d}_{λ} is $\overline{\mathbb{R}^{d}_{\lambda}} = \mathbb{R}^{d}_{\lambda} \cup \left\{ \sum_{i} r_{i}\omega^{a_{i}} \mid r_{i} \in \mathbb{R}, a_{i} \in \mathbf{No}_{c_{i}a} \text{ are coinitial} \right\}$

- If we speak about Cauchy gap, it is because we can avoid them. In particular we can have a complete field. The difference with a real case being that the sequence must have length the degree of the field, which is the coinitiality of the set of positive elements of the field.
- <u>Def</u> : Coinitial, Cofinal, Coinitiality, Cofinality.
- With Hahn series, we have a nice Cauchy completion

Quentin Guilmant

Cauchy completion

The Cauchy-completion of $\mathbb{R}^{\mu}_{\lambda}$ is

Proposition

Cauchy cuts and Cauchy completion

 \mathbb{K} is Cauchy-complete *iff* it has no non-trivial Cauchy gap. $\mathbb{R}^{\mu}_{\lambda} := \mathbb{R}^{No < \mu}_{\lambda}$ has a simple Cauchy-completion.

Proposition

$$\widetilde{\mathbb{R}^{\mu}_{\lambda}} = \mathbb{R}^{\mu}_{\lambda} \cup \left\{ \sum_{i < \lambda} r_{i} \omega^{a_{i}} \ \middle| \ r_{i} \in \mathbb{R}, \ a_{i} \in \mathbf{No}_{<\mu} \text{ are coinitial} \right\}$$

Non-trivial gaps in $\mathbb{R}^{\mu}_{\lambda}$ may be of the form :

For (a_i)_i not coinitial $\sum r_i \omega^{a_i}$ (Type 1) For $x \in \widetilde{\mathbb{R}^{\mu}_{\lambda}}$ and \mathcal{G} a gap of $No_{<\mu}$ $x + \omega^{\mathcal{G}}$ (Type 2)

Surreal numbers Cauchy cuts and Cauchy -26 K is Cauchy-complete iff it has no non-trivial Cauchy gap Computations with surreal numbers, existing me- $\mathbb{R}_{1}^{\mu} := \mathbb{R}_{1}^{Na_{1}\mu}$ has a simple Cauchy-completion 2019-11 The Cauchy-completion of $\mathbb{R}^{\mu}_{\lambda}$ is thods $\mathbb{R}^{\mu}_{\lambda} = \mathbb{R}^{\mu}_{\lambda} \cup \left\{ \sum_{i} r_{i} \omega^{a_{i}} | r_{i} \in \mathbb{R}, a_{i} \in \mathbf{No}_{<\mu} \text{ are coinitial} \right\}$ -Cauchy completion Non-trivial gaps in Rⁱⁿ may be of the form For (a;); not coinitial -Cauchy cuts and Cauchy completion For $x \in \mathbb{R}^{p}$, and \mathcal{G} a gap of No_{CA}

• If we speak about Cauchy gap, it is because we can avoid them. In particular we can have a complete field. The difference with a real case being that the sequence must have length the degree of the field, which is the coinitiality of the set of positive elements of the field.

(Type 2)

- Def : Coinitial, Cofinal, Coinitiality, Cofinality.
- With Hahn series, we have a nice Cauchy completion
- The remaining gaps are exactly what we expect. In particular, in type 1 gaps we just get rid of the case where the (a_i) s are coinitial.

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations Sub-structure and Hahn series Gaps in the surreal fields

Computations with surreal numbers, existing methods Cauchy completion

Computable Analysis

Problem of integration Motivation : Anal computing Some previous tri Handle the gaps,

new notion of compacity.

functions

Motivation for Cauchy-completion

Cauchy-completion is less powerful than Dedekind-completion (no non-trivial gap remaining).

Surreal numbers Computations with surreal numbers, existing methods Cauchy completion Motivation for Cauchy-completion Motivation for Cauchy-completion

Cauchy-completion is less powerful than Dedekind-completion (no non-trivial gap remaining).

Cauchy-completion and Dedekind-completion are different. It is because the field is not Archimedean. Dedekind-completion is much more powerful. So we have to give a motivation for the use of Cauchy-completion. Voir slide : bonne propriétés

Quentin Guilmant

Cauchy completion

Motivation for Cauchy-completion

Cauchy-completion is less powerful than Dedekind-completion (no non-trivial gap remaining). Many gaps remain in $\mathbb{R}^{\mu}_{\lambda}$ so why a Cauchy-completion?

- Surreal numbers -26 Computations with surreal numbers, existing me-H thods 2019-1 Cauchy completion
 - -Motivation for Cauchy-completion

Cauchy-completion is less powerful than Dedekind-completio (no non-trivial gap remaining). Many gaps remain in R^e so why a Cauchy-completion ?

Motivation for Cauchy-completion

Cauchy-completion and Dedekind-completion are different. It is because the field is not Archimedean. Dedekind-completion is much more powerful. So we have to give a motivation for the use of Cauchy-completion. Voir slide : bonne propriétés

Quentin Guilmant

Cauchy completion

Motivation for Cauchy-completion

Cauchy-completion is less powerful than Dedekind-completion (no non-trivial gap remaining). Many gaps remain in $\mathbb{R}^{\mu}_{\lambda}$ so why a Cauchy-completion?

• Cauchy-sequences (possibly with larger length) will converge (needed for some powerful theorem in analysis)

Surreal numbers -26 Computations with surreal numbers, existing me-H thods 2019-1 Cauchy completion

-Motivation for Cauchy-completion

Cauchy-completion is less powerful than Dedekind-completio (no non-trivial gap remaining). Many gaps remain in R^e so why a Cauchy-completion ? · Cauchy-sequences (possibly with larger length) will converge (needed for some powerful theorem in analysis

Motivation for Cauchy-completion

Cauchy-completion and Dedekind-completion are different. It is because the field is not Archimedean. Dedekind-completion is much more powerful. So we have to give a motivation for the use of Cauchy-completion. Voir slide : bonne propriétés

Quentin Guilmant

Cauchy completion

Motivation for Cauchy-completion

Cauchy-completion is less powerful than Dedekind-completion (no non-trivial gap remaining). Many gaps remain in $\mathbb{R}^{\mu}_{\lambda}$ so why a Cauchy-completion?

- Cauchy-sequences (possibly with larger length) will converge (needed for some powerful theorem in analysis)
- Computable analysis, Cauchy representations

Surreal numbers -26 Computations with surreal numbers, existing me-H thods 2019-1 Cauchy completion

-Motivation for Cauchy-completion

Cauchy-completion is less powerful than Dedekind-completio (no non-trivial gap remaining). Many gaps remain in R^e so why a Cauchy-completion ? · Cauchy-sequences (possibly with larger length) will converge (needed for some powerful theorem in analysis Computable analysis. Cauchy representations

Cauchy-completion and Dedekind-completion are different. It is because the field is not Archimedean. Dedekind-completion is much more powerful. So we have to give a motivation for the use of Cauchy-completion. Voir slide : bonne propriétés

Quentin Guilmant

Cauchy completion

Motivation for Cauchy-completion

Cauchy-completion is less powerful than Dedekind-completion (no non-trivial gap remaining). Many gaps remain in $\mathbb{R}^{\mu}_{\lambda}$ so why a Cauchy-completion?

- Cauchy-sequences (possibly with larger length) will converge (needed for some powerful theorem in analysis)
- Computable analysis, Cauchy representations
- Better generalization of \mathbb{R}

Surreal numbers -26 Computations with surreal numbers, existing me-H thods 2019-1

- Cauchy completion
 - -Motivation for Cauchy-completion

Cauchy-completion is less powerful than Dedekind-completion (no non-trivial gap remaining). Many gaps remain in R^e so why a Cauchy-completion ? · Cauchy-sequences (possibly with larger length) will converge (needed for some powerful theorem in analysis) · Computable analysis. Cauchy representations Better generalization of E

Motivation for Cauchy-completion

Cauchy-completion and Dedekind-completion are different. It is because the field is not Archimedean. Dedekind-completion is much more powerful. So we have to give a motivation for the use of Cauchy-completion. Voir slide : bonne propriétés

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

fields.

Definitions and operations

Sub-structure and Hahn series

Computations with surreal numbers, existing methods

Cauchy completion

Computable Analysis over surreal fields

Problem of integration

Motivation : Analog computing

Some previous tries

Handle the gaps, a new notion of compacity.

Strongly continuo functions

Perspectiv

Intervals representation (1/2)

Surreal numbers Computations with surreal numbers, existing methods Computable Analysis over surreal fields Intervals representation (1/2)

- $\mathbb{R}^{\mu}_{\lambda}$ admits a notation over $\{0,1\}^{<\lambda}$. It is dense in $\widetilde{\mathbb{R}^{\mu}_{\lambda}}$.
 - Intervals of $\mathbb{R}^{\mu}_{\lambda}$ with bounds in $\mathbb{R}^{\mu}_{\lambda}$ have a notation.
 - Give $\widetilde{\mathbb{R}^{\mu}_{\lambda}}$ a structure of an effective (topological) space.

Intervals representation (1/2)

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations

Sub-structure and Hahn series

fields.

Computations with surreal numbers, existing methods

Cauchy completion

Computable Analysis over surreal fields

Problem of integration

Motivation : Analo computing

Some previous tries

Handle the gaps, new notion of compacity

Strongly continuo functions

Perspectiv

Interlude : Effective space

Definition (λ -effective space)

A λ -effective space is a triplet $\mathcal{M} = (M, \sigma, \nu)$ with M a set, $\sigma \subseteq 2^M$ a collection of subsets of M such that

 $\begin{array}{l} x = y \quad \Leftrightarrow \quad \{A \in \sigma \mid x \in A\} = \{A \in \sigma \mid y \in A\} \\ \text{and } \nu :\subseteq \{0, 1\}^{<\lambda} \to \sigma \text{ is a notation.} \end{array}$

Surreal numbers Computations with surreal numbers, existing methods Computable Analysis over surreal fields

Computable Analysis over surreal fields

Interlude : Effective space

-26

H

2019-1

Interlude : Effective space

Definition (λ -effective space)

A λ -effective space is a triplet $\mathcal{M}=(M,\sigma,\nu)$ with M a set, $\sigma\subseteq 2^M$ a collection of subsets of M such that $x=y \quad \Leftrightarrow \quad \{A\in\sigma\mid x\in A\}=\{A\in\sigma\mid y\in A\} \text{ and }\nu:\subseteq\{0,1\}^{<\lambda}\to\sigma \text{ is a notation}.$

Quentin Guilmant

Computable Analysis over surreal fields

Interlude : Effective space

Definition (λ -effective space)

A λ -effective space is a triplet $\mathcal{M} = (M, \sigma, \nu)$ with M a set, $\sigma \subset 2^M$ a collection of subsets of M such that

 $x = y \quad \Leftrightarrow \quad \{A \in \sigma \mid x \in A\} = \{A \in \sigma \mid y \in A\}$ and $\nu :\subseteq \{0,1\}^{<\lambda} \to \sigma$ is a notation.

Definition

- The standard topology $\tau_{\mathcal{M}}$ is the topology induced by σ
- The standard representation $\delta_{\mathcal{M}} :\subseteq \{0,1\}^{\lambda} \to M$ is given by

$$\delta_{\mathcal{M}}(p) = x \iff \{A \in \sigma \mid x \in A\} = \{\nu(w) \mid \iota(w) \sqsubset p\}$$

Surreal numbers -26 Computations with surreal numbers, existing me-H thods 2019-1 -Computable Analysis over surreal fields

-Interlude : Effective space

Interlude : Effective space

Definition (λ -effective space)

A λ -effective space is a triplet $\mathcal{M} = (M, \sigma, \nu)$ with M a set. $\sigma \subseteq 2^M$ a collection of subsets of M such that $x = y \iff \{A \in \sigma \mid x \in A\} = \{A \in \sigma \mid y \in A\}$ and $\nu :\subseteq \{0, 1\}^{<\lambda} \rightarrow \sigma$ is a notation.

Definitio

 The standard topology \u03c6_M is the topology induced by \u03c6 The standard representation δ_M :⊂ {0,1}^λ → M is given

 $\delta_M(p) = x \iff \{A \in \sigma \mid x \in A\} = \{\nu(w) \mid \iota(w) \sqsubset p\}$

Quentin Guilmant

- fields.

Computable Analysis over surreal fields

Motivation : Analog

Intervals representation (2/2)

.

Surreal numbers -26 Computations with surreal numbers, existing me-2019-11thods -Computable Analysis over surreal fields \square Intervals representation (2/2)

Intervals representation (2/2)

 Take σ = {] a; b [| a, b ∈ ℝ^p₁} and ν a notation over σ. (ℝⁿ_λ, σ, ν) is λ-effective TW is the interval topology.

• Take
$$\sigma=\{\,]\,a$$
 ; $b\,[\,\mid\,a,b\in\mathbb{R}^\mu_\lambda\}$ and u a notation over σ .

- $(\mathbb{R}^{\mu}_{\lambda}, \sigma, \nu)$ is λ -effective
- $\tau_{\mathcal{M}}$ is the interval topology.

Quentin

fields.

Computable Analysis

over surreal fields

Guilmant

Cauchy representation

Surreal numbers -26 Computations with surreal numbers, existing me-Definition $\delta_{\mathcal{C}}(p) = x \quad \Leftrightarrow \begin{cases} p = \iota(w_0)\iota(w_1) \cdots \\ (\iota(w_\alpha))_{\alpha \in \lambda} \text{ is quickly convergent to } x \end{cases}$ 2019-11 thods $\forall \alpha \leq \beta < \lambda$ $|\nu(\mathbf{w}_{\alpha}) - \nu(\mathbf{w}_{\beta})| \leq \frac{1}{1 + 2}$ -Computable Analysis over surreal fields Proposition (Galeotti [5]) δ_M and δ_C are equivalent -Cauchy representation

Cauchy representation

Definition

 $\delta_{\mathcal{C}}(p) = x \quad \Leftrightarrow \begin{cases} p = \iota(w_0)\iota(w_1)\cdots\\(\nu(w_\alpha))_{\alpha<\lambda} \text{ is quickly convergent to } x \end{cases}$ $\forall \alpha \leq \beta < \lambda \qquad |\nu(w_\alpha) - \nu(w_\beta)| \leq \frac{1}{\alpha+1}$

Proposition (Galeotti [5])

 $\delta_{\mathcal{M}}$ and $\delta_{\mathcal{C}}$ are equivalent.

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations

Sub-structure and Hahn series Gaps in the surreal fields.

4 Problem of integration

Computation with surreal numbers.

existing nethods

Computable Analysis over surreal fields

Problem of integration

Motivation : Analog computing

some previous tries

new notion of compacity.

Strongly continuous functions

Perspective

Outline

O Problem of integration

Outline

Quentin Guilmant

fields.

Motivation : Analog computing

Analog computers over the reals

и

v

V -

+



Analog computers over the reals

″_____ + − *u* + *v* k k $u \rightarrow u \rightarrow v$ $u \rightarrow v$ $u \rightarrow f_0^t u dv$

Definition

k

 \times

U

v

 $- u \times v$

-k



u + v

Quentin Guilmant

Introduction Numbers

Surreal Numbers

Definitions and operations

Sub-structure and Hahn series

Gaps in the surrea fields.

Computations with surreal numbers, existing methods Cauchy completion Computable Analysis

Problem of integration

Motivation : Analog computing

Some previous tries

Handle the gaps, new notion of compacity.

Strongly continuou functions

Perspective

Computing sinh

Surreal numbers Problem of integration Motivation : Analog computing Computing sinh



Computing sinh



Quentin Guilmant

Introduction Numbers

Surreal Numbers

Definitions and operations

Sub-structure and Hahn series

Gaps in the surrest fields.

Computations with surreal numbers, existing methods Cauchy completion

Computable Analysis over surreal fields

Problem of integration

Motivation : Analog computing

Some previous tries

new notion of compacity.

Strongly continuou functions

Perspectiv

Computing sinh

Surreal numbers Problem of integration Motivation : Analog computing Computing sinh



Computing sinh

GPACs correspond to (vectorial) pODE.



• GPACs correspond to (vectorial) pODE.

Quentin Guilmant

Introduction Numbers

Surreal Numbers

Definitions and operations

Sub-structure and Hahn series

Gaps in the surreative fields.

Computations with surreal numbers, existing methods Cauchy completion Computable Analysis

Problem of integration

Motivation : Analog computing

Handle the gaps, a new notion of

Strongly continue

Perspectiv

Computing sinh

Surreal numbers Problem of integration Motivation : Analog computing Computing sinh

Computing sinh

GPACs correspond to (vectorial) pODE.
 Polynomials are continuous and locally Lipschitz



- GPACs correspond to (vectorial) pODE.
- Polynomials are continuous and locally Lipschitz.

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations

Sub-structure and Hahn series

Gaps in the surrea fields.

Computation

numbers, existing

Cauchy completion Computable Analys

Problem of

Motivation : Analog computing

Handle the gaps, a new notion of compacity. A solution exists

Strongly continuo functions

Perspectiv

Dependences for Picard-Lindelöf (a.k.a Cauchy-Lipschitz) Theorem

Surreal numbers Problem of integration Motivation : Analog computing Dependences for Picard-Lindelöf (a.k.a Cauchy-Lipschitz) Theorem

The solution is unique
Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations

Sub-structure and Hahn series

Gaps in the surrea fields.

computations with surreal numbers, existing methods Cauchy completion

Computable Analysis over surreal fields

Problem of integration

Motivation : Analog computing

Handle the gaps, a new notion of compacity.

Strongly continuou functions

Perspecti



Dependences for Picard-Lindelöf (a.k.a Cauchy-Lipschitz) Theorem

The solution is unique

Surreal numbers Problem of integration Motivation : Analog computing Dependences for Picard-Lindelöf (a.k.a Cauchy-Lipschitz) Theorem Dependences for Picard-Lindelöf (a.k.a Cauchy-Lipschitz) Theorem

Banach Integral operator	
A solution exists	The solution is unique

Quentin Guilmant

Motivation : Analog computing

Integral operator Banach A solution exists



Dependences for Picard-Lindelöf

(a.k.a Cauchy-Lipschitz) Theorem

Surreal numbers 11-26 Problem of integration -Motivation : Analog computing 2019-1 -Dependences for Picard-Lindelöf (a.k.a Cauchy-Lipschitz) Theorem

Dependences for Picard-Lindelöf (a.k.a Cauchy-Lipschitz) Theorem

Banach Integral op	Grönwall
A solution exists	The solution is unique

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations

Sub-structure and Hahn series

Gaps in the surres fields.

Computations with surreal numbers, existing methods Cauchy completion Computable Analysis over surreal fields

Problem of ntegration

Motivation : Analog computing

Handle the gaps, a new notion of compacity.

Strongly continuou functions

Perspectiv

Banach Integral operator Grönwall A solution exists The solution is unique

Dependences for Picard-Lindelöf

(a.k.a Cauchy-Lipschitz) Theorem

Sign of the derivative \Leftrightarrow variations

Surreal numbers Problem of integration Motivation : Analog computing Dependences for Picard-Lindelöf (a.k.a Cauchy-Lipschitz) Theorem Dependences for Picard-Lindelöf (a.k.a Cauchy-Lipschitz) Theorem

Sign	of the derivative ++ variations
Banach Integral operator	Grönwall
A solution exists	The solution is unique

Quentin Guilmant

Banach

Motivation : Analog computing

Dependences for Picard-Lindelöf (a.k.a Cauchy-Lipschitz) Theorem

Surreal numbers Dependences for Picard-Lindelö (a.k.a Cauchy-Lipschitz) Theorem 11-26 Problem of integration -Motivation : Analog computing 2019-1 Sign of the derivative ++ variations -Dependences for Picard-Lindelöf (a.k.a Banach Integral operator The solution is uniqu Cauchy-Lipschitz) Theorem A solution exists

MVT



Quentin Guilmant

Motivation : Analog computing

Dependences for Picard-Lindelöf (a.k.a Cauchy-Lipschitz) Theorem

Surreal numbers Dependences for Picard-Lindelöf (a.k.a Cauchy-Lipschitz) Theorem 11-26 Problem of integration -Motivation : Analog computing 2019-1 -Dependences for Picard-Lindelöf (a.k.a Banach Integral operator Cauchy-Lipschitz) Theorem A solution exists

MV

Sign of the derivative ++ variations

The solution is uniqu



Quentin Guilmant

Motivation : Analog computing

Dependences for Picard-Lindelöf (a.k.a Cauchy-Lipschitz) Theorem



Surreal numbers Dependences for Picard-Lindelöf (a.k.a Cauchy-Lipschitz) Theorem 11-26 Problem of integration -Motivation : Analog computing 2019-1 Sign of the derivative ++ variations -Dependences for Picard-Lindelöf (a.k.a Banach Int A solution exist Integral operator Cauchy-Lipschitz) Theorem The solution is uniqu

MVT

Grónwall

Quentin Guilmant

Motivation : Analog computing

Dependences for Picard-Lindelöf (a.k.a Cauchy-Lipschitz) Theorem





MVT

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations Sub-structure and Hahn series Gaps in the surreal fields.

Computations with surreal numbers, existing nethods Cauchy completion Computable Analysis over surreal fields

Problem of integration

Motivation : Analog computing

Some previous tries Handle the gaps, a new notion of compacity.

Strongly continuou functions

Perspectiv

Dependences for Picard-Lindelöf (a.k.a Cauchy-Lipschitz) Theorem



Surreal numbers Problem of integration Motivation : Analog computing Dependences for Picard-Lindelöf (a.k.a Cauchy-Lipschitz) Theorem

Quentin Guilmant

Motivation : Analog computing

Dependences for Picard-Lindelöf (a.k.a Cauchy-Lipschitz) Theorem



Surreal numbers Dependences for Picard-Lindelö (a.k.a Cauchy-Lipschitz) Theorem 11-26 Problem of integration -Motivation : Analog computing 2019-1 Sign of the derivative ++ variations -Dependences for Picard-Lindelöf (a.k.a Grönwall The solution is unique Cauchy-Lipschitz) Theorem A solution exist

MVT

Quentin Guilmant

Motivation : Analog computing

Dependences for Picard-Lindelöf (a.k.a Cauchy-Lipschitz) Theorem



Surreal numbers Dependences for Picard-Lindelö (a.k.a Cauchy-Lipschitz) Theorem 11-26 Problem of integration -Motivation : Analog computing 2019-1 Sign of the derivative ++ variations -Dependences for Picard-Lindelöf (a.k.a Cauchy-Lipschitz) Theorem A solution exis

MVT

Grönwall The solution is unique

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations Sub-structure and Hahn series Gaps in the surreal fields.

Computations with surreal numbers, existing nethods Cauchy completion Computable Analysis over surreal fields

Problem of integration

Motivation : Analog computing

Some previous tries Handle the gaps, a new notion of compacity.

Strongly continuo functions

Perspectiv

Dependences for Picard-Lindelöf (a.k.a Cauchy-Lipschitz) Theorem



Surreal numbers Problem of integration Motivation : Analog computing Dependences for Picard-Lindelöf (a.k.a Cauchy-Lipschitz) Theorem



Sign of the derivative +> variations Banach Integral operator Gréewall A solution exists The solution is unique

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations Sub-structure and Hahn series Gaps in the surreal fields.

Computations with surreal numbers, existing methods Cauchy completion Computable Analysis

Problem of integration

Motivation : Analog computing

Some previous tries

Handle the gaps, a new notion of compacity.

Strongly contin functions

Perspectiv

Generalized Riemann Sums

Definition (Rubinstein-Salzedo, Swaminathan [7]) Let $\mathbb{K} \subseteq \mathbf{No}$, $\mathcal{F} \subseteq \bigcup_{n} (\mathbb{K}^{n} \to \mathbb{K})$ and \mathscr{F} be its $(+, \times, \circ)$ -closure. Let $f : [a; b] \to \mathbb{K}$ continuous. If there is $g \in \mathscr{F}$ such that $\forall n \in \mathbb{N} \ \forall a \leq c \leq d \leq b \quad g(n, c, d) = \sum_{i=0}^{n} \frac{d-c}{n} f(c+i\frac{d-c}{n})$ Then for $\alpha \in \mathbb{K}$ an ordinal $g(\alpha, a, b)$ is called the $(\mathbb{K}, \mathcal{F})$ -Riemann sum of f of order α . Surreal numbers Problem of integration Some previous tries Generalized Riemann Sums

Generalized Riemann Sums

Definition (Rubinettin-Salvedo, Swaminathan [7]) Let $\mathbb{K} \subseteq \mathbf{N} a$, $F \subseteq \bigcup \{\mathbb{K}^n \to \mathbb{K}\}$ and \mathcal{F} be its (+, x, o)-closure. Let $f : [a, b] \to \mathbb{K}$ continuous. If there is $g \in \mathcal{F}$ such that $\forall n \in \mathbb{N} \ \forall a \le c \le d \le b$, $g(n, c, d) = \frac{1}{n} \frac{d-c}{d-n} f(c + i \frac{d-c}{n})$ Then for $n \in \mathbb{K}$ an ordinal g(n, a, b) is called the (K, F)-Riverana sum of f of order n.

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations Sub-structure and Hahn series Gaps in the surreal fields

Computations with surreal numbers, existing methods Cauchy completion Computable Analysis over surreal fields

Problem of ntegration Motivation : Analog computing

Some previous tries

Handle the gaps, a new notion of compacity.

Strongly continuou functions

Perspectives

Generalized Riemann Sums

Definition (Rubinstein-Salzedo, Swaminathan [7]) Let $\mathbb{K} \subseteq \mathbf{No}$, $\mathcal{F} \subseteq \bigcup_{n} (\mathbb{K}^{n} \to \mathbb{K})$ and \mathscr{F} be its $(+, \times, \circ)$ -closure. Let $f : [a; b] \to \mathbb{K}$ continuous. If there is $g \in \mathscr{F}$ such that $\forall n \in \mathbb{N} \ \forall a \leq c \leq d \leq b \quad g(n, c, d) = \sum_{i=0}^{n} \frac{d-c}{n} f(c+i\frac{d-c}{n})$ Then for $\alpha \in \mathbb{K}$ an ordinal $g(\alpha, a, b)$ is called the $(\mathbb{K}, \mathcal{F})$ -Riemann sum of f of order α .

Issues

Is g unique? Do the properties of g for $n \in \mathbb{N}$ transfer to ordinal number?

Surreal numbers Problem of integration Some previous tries Generalized Riemann Sums

Generalized Riemann Sums

Definition (Robinstein-Schedo, Suominathan [7]) Let $\mathbb{K} \subseteq \mathbf{N}0$, $\mathcal{F} \subseteq \bigcup_{i=1}^{d} \mathbb{K}^n \to \mathbb{K}$) and \mathcal{F} be its $(+, \times, \circ)$ -closure. Let $f: [a; b] \to \mathbb{K}$ continuous. If there is $g \in \mathcal{F}$ such that $\forall n \in \mathbb{N} \forall a \leq c \leq d \leq b$ $g(n, c, d) \to \sum_{i=1}^{d} \frac{d}{n-1} (c+i\frac{d-c}{n})$ Then for $n \in \mathbb{K}$ as ordinal g(n, a, b) is called the $(\mathbb{K}, \mathcal{F})$ -formant us of f of oder a.

Is g unique ? Do the properties of g for $n \in \mathbb{N}$ transfer to ordinal number ?

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations Sub-structure and Hahn series Gaps in the surreal fields.

Computations with surreal numbers, existing methods Cauchy completion Computable Analysis

Problem of integration

Motivation : Analog computing

Some previous tries

Handle the gaps, a new notion of compacity.

functions

Perspectiv

Fornasiero [4] : Give an inductive definition for the integration operator. Works with "genetic definitions".

Genetic definition

Surreal numbers Problem of integration Some previous tries Genetic definition

Fornasiero [4] : Give an inductive definition for the integration operator. Works with "genetic definitions".

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations Sub-structure and

Hahn series Gaps in the surres fields.

Computation with surreal numbers, existing

nethods Cauchy completion Computable Analys

Problem of integration

Motivation : Ana computing

Some previous tries

Handle the gaps, a new notion of compacity.

Strongly continuous functions

Perspectives

Comparison

Surreal numbers Problem of integration Some previous tries Comparison

	Fornasiero	Rubinstein
∫ ₂ ⁰ cdt	 	×
Linearity	 	
Definite	V	Not studied
	("Recursive functions")	
$\int_{a}^{b} f = F(b) - F(a)$	×	 ✓
TFA	(not unique)	(not unique)
Rolle	V	 (strongly)
		continuous)
IAF	Non étudiée	
Integration	Genetic "recursive"	Riemann sums
hundhesis	(nathological)	converse

Comparison

	Fornasiero	Rubinstein
$\int_{a}^{b} c dt$	 ✓ 	 ✓
Linearity	 ✓ 	 ✓
Definite		Not studied
	("Recursive functions")	
$\int_a^b f = F(b) - F(a)$	×	 Image: A set of the set of the
TFA	✓(not unique)	✓(not unique)
Rolle	 ✓ 	🖌 (strongly
		continuous)
IAF	Non étudiée	~
Integration	Genetic "recursive"	Riemann sums
hypothesis	(pathological)	converge

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations Sub-structure and Hahn series Gaps in the surreal fields.

Computations with surreal numbers, existing methods Cauchy completion Computable Analysis over surreal fields

Problem of integration

Motivation : Analog computing

Some previous tries

Handle the gaps, a new notion of compacity.

Strongly continuou functions

Perspectiv

Strongly compact subsets

Idea

Gaps are the problem. Strongly compact sets must have a Lebesgue-Borel-like property but must be blind to gaps.

Surreal numbers Problem of integration Handle the gaps, a new notion of compacity. Strongly compact subsets

11-26

2019-0



Gaps are the problem. Strongly compact sets must have a Lebesgue-Borel-like property but must be blind to gaps.

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations Sub-structure and Hahn series Gaps in the surreal fielde

Computations with surreal numbers, xisting nethods Cauchy completion Computable Analysis over surreal fields

Problem of integration Motivation : Analog computing Some previous tries Handle the gaps, a new notion of compacity.

functions

Perspective

Strongly compact subsets

ldea

Gaps are the problem. Strongly compact sets must have a Lebesgue-Borel-like property but must be blind to gaps.

Definition ((λ, μ)-strongly-compact set)

If \mathcal{X} is a set of open intervals of $\mathbb{R}^{\mu}_{\lambda}$, let $\mathcal{B}(\mathcal{X})$ the set of the bounds of theses intervals. Now, a subset $X \subseteq \mathbb{R}^{\mu}_{\lambda}$ is said (λ, μ) -strongly-compact if for any covering \mathcal{X} of X by open intervals with no non-trivial partition $L \cup R = \mathcal{B}(\mathcal{X})$ such that L < R and $[L \mid R]$ is a gap in $\mathbb{R}^{\mu}_{\lambda}$, there is a finite sub-covering.

Surreal numbers Problem of integration Handle the gaps, a new notion of compacity. Strongly compact subsets

Strongly compact subsets

Gaps are the problem. Strongly compact sets must have a Lebesgue-Borel-like property but must be blind to gaps.

Definition ((λ, μ)-strongly-compact set)

If X is a set of open intervals of $\overline{\mathbb{R}}^{T}_{A}$, let B(X) the set of the bounds of theses intervals. Now, a subset $X \subseteq \mathbb{R}^{T}_{A}$ is said (A, μ) -strongly-compact if for any covering X of X by open intervals with no non-trivial partition $L \cup R = B(X)$ such that L < R and (L : R) is a gap in \mathbb{R}^{T}_{A} , there is a finite sub-covering

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations Sub-structure and Hahn series Gaps in the surreal fields

computations with surreal umbers, xisting nethods Cauchy completion Computable Analysis wer surreal fields

Problem of integration Motivation : Analog computing Some previous tries Handle the gaps, a new notion of compacity.

Perspectives

Strongly compact subsets

ldea

Gaps are the problem. Strongly compact sets must have a Lebesgue-Borel-like property but must be blind to gaps.

Definition ((λ, μ)-strongly-compact set)

If \mathcal{X} is a set of open intervals of $\mathbb{R}^{\mu}_{\lambda}$, let $\mathcal{B}(\mathcal{X})$ the set of the bounds of theses intervals. Now, a subset $X \subseteq \widetilde{\mathbb{R}^{\mu}_{\lambda}}$ is said (λ, μ) -strongly-compact if for any covering \mathcal{X} of X by open intervals with no non-trivial partition $L \cup R = \mathcal{B}(\mathcal{X})$ such that L < R and $[L \mid R]$ is a gap in $\widetilde{\mathbb{R}^{\mu}_{\lambda}}$, there is a finite sub-covering. In other words : Every no-gap-showing covering admits a finite sub-covering.

Surreal numbers Problem of integration Handle the gaps, a new notion of compacity. Strongly compact subsets

Strongly compact subsets

Gaps are the problem. Strongly compact sets must have a Lebesgue-Borel-like property but must be blind to gaps.

Definition ((λ, μ)-strongly-compact set)

If X is a set of open intervals of $S_{n_1}^{(2)}$ test S(X) the set of the bounds of these intervals. Now, a subset $X \subset S_{n_1}^{(2)}$ is said (λ, μ) -strongly-compact if for any covering X of X by open intervals with no non-trivial partitions $L \cup R \to B(X)$ such that L < R and $(L \mid R)$ is a gap in $S_{n_1}^{(2)}$, there is a finite sub-covering. In other model, if ways no gap-showing covering admits a finite sub-covering.

Quentin Guilmant

Introduction : Numbers

rreal mbers

Theorem

of the following is true

Definitions and operations

Sub-structure and Hahn series Gaps in the surreal fields.

Computations with surreal numbers, existing methods Cauchy completion Computable Analysis over surreal fields

Problem of integration

Motivation : Analog computing

Some previous tries

Handle the gaps, a new notion of compacity.

Strongly continuou functions

Perspecti

Alternative definitions

 (λ,μ) -strongly-compact set the subsets $X\subseteq \widetilde{\mathbb{R}^{\mu}_{\lambda}}$ such that one

Surreal numbers Problem of integration Handle the gaps, a new notion of compacity. Alternative definitions

Theorem (λ, μ) -strongly-compact set the subsets $X \subseteq \overline{\mathbb{R}^{\mu}_{\lambda}}$ such that one of the following is true

Alternative definitions

Quentin Guilmant

Theorem

of the following is true

(resp. sup $Z = \inf Y$)

Handle the gaps, a new notion of compacity.

Alternative definitions

 (λ, μ) -strongly-compact set the subsets $X \subseteq \widetilde{\mathbb{R}^{\mu}}$ such that one

• X is closed and for any $Y \subseteq X$ such that sup Y (resp. inf Y) is a gap, there is $Z \subseteq X$ such that inf $Z = \sup Y$

Surreal numbers 11-26 Problem of integration -Handle the gaps, a new notion of compacity. 2019-1 -Alternative definitions

 (λ, μ) -strongly-compact set the subsets $X \subseteq \mathbb{R}^{\mu}_{\lambda}$ such that one of the following is true • X is closed and for any $Y \subseteq X$ such that sup Y (resp inf Y) is a gap, there is $Z \subseteq X$ such that inf $Z = \sup Y$ (resp. sup Z = inf Y)

Alternative definitions

Quentin Guilmant

Introduction : Numbers

rreal mbers

Theorem

of the following is true

(resp. sup $Z = \inf Y$)

a finite sub-covering.

Definitions and operations

Sub-structure and Hahn series Gans in the surreal

Computations with surreal numbers, existing methods Cauchy completion Computable Analysis

Problem of integration Motivation : Analog

Some previous tries

Handle the gaps, a new notion of compacity.

Strongly continuou functions

Perspectiv

Alternative definitions

 (λ, μ) -strongly-compact set the subsets $X \subseteq \mathbb{R}^{\mu}_{\lambda}$ such that one

X is closed and for any Y ⊆ X such that sup Y (resp. inf Y) is a gap, there is Z ⊆ X such that inf Z = sup Y

• Every covering no-gap-showing \mathcal{X} of X by open sets admits

2019-11

-26

Surreal numbers Problem of integration Handle the gaps, a new notion of compacity. Alternative definitions

 $\label{eq:states} \begin{array}{l} \hline \textbf{Theorem} \\ (\lambda, \mu) \mbox{-transformation} X \subseteq \overline{\mathbb{R}^n_\lambda} \mbox{ such that one of the following is true} \\ & X \mbox{ is closed and for any } Y \subseteq X \mbox{ such that sup } Y \mbox{ (resp. inf } Y) \mbox{ is a gap, three is } Z \subseteq X \mbox{ such that inf } Z = \mbox{ sup } Y \mbox{ (resp. inf } Y) \mbox{ is a gap, three is } Z \subseteq X \mbox{ such that inf } Z = \mbox{ sup } Y \mbox{ (resp. inf } Y) \mbox{ is a gap, three is } Z \subseteq X \mbox{ such that inf } Z = \mbox{ sup } Y \mbox{ (resp. inf } Y) \mbox{ is a gap, three is } Z \subseteq X \mbox{ such that inf } Z = \mbox{ sup } Y \mbox{ (resp. inf } Y) \mbox{ is a gap, three is } Z \subseteq X \mbox{ such that inf } Z = \mbox{ sup } Y \mbox{ (resp. inf } Y) \mbox{ is a gap, three is } Z \in X \mbox{ such that inf } Z = \mbox{ sup } Y \mbox{ (resp. inf } Y) \mbox{ is a gap, three is } Z \in X \mbox{ such that inf } Z = \mbox{ sup } Y \mbox{ (resp. inf } Y) \mbox{ is a gap, three is } Z \in X \mbox{ such that inf } Z = \mbox{ sup } Y \mbox{ (resp. inf } Y) \mbox{ is a gap, three is } Z \in X \mbox{ such that inf } Z = \mbox{ sup } Y \mbox{ (resp. inf } Y) \mbox{ is a gap, three is } Z \in X \mbox{ sup } Y \mbox{ sup } X \mbox{ sup } Y \mbox{ sup } Y \mbox{ sup } X \mbox{ sup } Y \mbox{ sup } X \mbox{ sup } Y \mbox{ sup } Y \mbox{ sup } Y \mbox{ sup } Y \mbox{ sup } X \mbox{ sup } Y \mbox{ sup }$

Alternative definitions

 Every covering no-gap-showing X of X by open sets admit a finite sub-covering.

With open sets : open sets are union of (strongly)-disjoint open intervals. You have to consider the bounds of such intervals to say that a covering is or not gap-showing.

Quentin Guilmant

Introduction : Numbers

rreal mbers

Theorem

of the following is true

(resp. sup $Z = \inf Y$)

a finite sub-covering.

Definitions and operations

- Sub-structure and Hahn series Gaps in the surreal
- Computations with surreal numbers, existing methods Cauchy completion Computable Analysis

Problem of Integration

computing

Handle the gaps, a new notion of

compacity. Strongly continuou

Perspectives

Alternative definitions

 (λ, μ) -strongly-compact set the subsets $X \subseteq \mathbb{R}^{\mu}_{\lambda}$ such that one

X is closed and for any Y ⊆ X such that sup Y (resp. inf Y) is a gap, there is Z ⊆ X such that inf Z = sup Y

• Every covering no-gap-showing \mathcal{X} of X by open sets admits

• Gaps $[L \mid R]$ are allowed if there is $Y \in \mathcal{X}$ and $I \subseteq Y$ and

open interval such that $\inf I < [L \mid R] < \sup I$

2019-11.

-26

Surreal numbers

Problem of integration
Handle the gaps, a new notion of compacity.
Alternative definitions
For the gaps and the set of the set of

a finite sub-covering.
 Gaps [L | R] are allowed if there is Y ∈ X and I ⊆ Y as open interval such that inf I < [L | R] < sup I

With open sets : open sets are union of (strongly)-disjoint open intervals. You have to consider the bounds of such intervals to say that a covering is or not gap-showing.

Alternative definitions

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations

Sub-structure and Hahn series Gaps in the surreal fields.

Computations with surreal numbers, existing methods Cauchy completion Computable Analysis over surreal fields

Problem of integration

Motivation : Analog computing

Some previous tries

Handle the gaps, a new notion of compacity.

Strongly continuou functions

Perspecti

An example

Surreal numbers Problem of integration Handle the gaps, a new notion of compacity. An example

a G b



Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations

Sub-structure and Hahn series Gaps in the surreal fields.

Computations with surreal numbers, existing methods Cauchy completion

Computable Analysis over surreal fields

Problem of integration

Motivation : Analog computing

Some previous tries

Handle the gaps, a new notion of compacity.

Strongly continuou functions

Perspective

An example

Surreal numbers Problem of integration Handle the gaps, a new notion of compacity. An example

Ą		G	ь	
,	13	'AL	,	4

An example



Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations

Sub-structure and Hahn series Gaps in the surreal fields.

Computations with surreal numbers, existing methods Cauchy completion

Computable Analys over surreal fields

Problem of integration

Motivation : Analog computing

Some previous tries

Handle the gaps, a new notion of compacity.

Strongly continuou functions

Perspectiv

An example

Surreal numbers Problem of integration Handle the gaps, a new notion of compacity.

÷	G	ь
,i	1200-12	<u> </u>

An example



Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations

Sub-structure and Hahn series Gaps in the surreal fields.

Computations with surreal numbers, existing methods Cauchy completion Computable Analysis

over surreal fields

integration

Motivation : Analog computing

Some previous tries

Handle the gaps, a new notion of compacity.

Strongly continuous functions

Perspectiv

An example

Surreal numbers Problem of integration Handle the gaps, a new notion of compacity.

ŕ	G	ь	
,,	5 <u>6</u> 47	, ·	

An example



Quentin Guilmant

Introduction : Numbers

Definitions and operations Sub-structure and Hahn series

Computations with surreal numbers, existing methods Cauchy completion Computable Analysis over surreal fields

Problem of integration Motivation : Analog computing Some previous tries Handle the gaps, a new notion of compacity.

Strongly continuous functions

Strongly continuous functions

Definition

A function $f: \mathbb{R}^{\mu}_{\lambda} \to \mathbb{R}^{\mu}_{\lambda}$ is said to be (λ, μ) -strongly-continuous if it is continuous and for any non-trivial gap $G = [L \mid R]$ of $\mathbb{R}^{\mu}_{\lambda}$ either f has a limit in G that is reached on any neighborhood of G or there is a non-trivial gap $H = [A \mid B]$ such that for any neighborhood J of H there is a neighborhood I of G such that

 $x \in I \implies f(x) \in J$ $f(I) \cap]H; \sup J[\neq \emptyset \quad \text{and} \quad f(I) \cap]\inf J; H[\neq \emptyset$ Surreal numbers Problem of integration Strongly continuous functions Strongly continuous functions

Strongly continuous functions

on ____ no

A function $f : \mathbb{R}_{q}^{2} \to \mathbb{R}_{q}^{2}$ is said to be $(L_{q})_{q}$ -transport continuous if H is continuous and for any mon-trivial gap $G = \lfloor L \mid B \mid G \mathbb{R}_{q}^{2}$ sticht f has a timi is G that is reached on any singhborhood G of or there is a non-trivial gap $H = \lfloor A \mid B \rfloor$ such that for any neighborhood J of H there is a neighborhood J of G such that $x \in I \to \sigma(R_{q}) \in J$ $(f(n) \mid F| : upg) \mid f \in G$ and f this

Quentin Guilmant

Introduction Numbers

Surreal Numbers

Definitions and operations Sub-structure and Hahn series Gaps in the surreal

Computations vith surreal numbers, existing nethods Cauchy completion Computable Analysis over surreal fields

Yroblem of ntegration Motivation : Analog computing Some previous tries Handle the gaps, a new notion of compacity.

Strongly continuous functions

Perspectives

Basic analysis of strongly continuous functions



Basic analysis of strongly continuous functions

Proposition (Intermediate value theorem)

Let $f : \mathbb{R}^n_+ \rightarrow \mathbb{R}^n_+$ be (λ, μ) -strongly-continuous. Assume $f(a) \leq f(b)$. Thus for all $y \in [\min(f(a), f(b)); \max(f(a), f(b))]$ there is $a \leq c \leq b$ such that f(c) = y.

Theorem (Extreme Value Theorem)

Let $f : \overline{\mathbb{R}}_{\lambda}^{\mu} \to \overline{\mathbb{R}}_{\lambda}^{n}$ be a (λ, μ) -strongly-continuous function. Let $X \subseteq \overline{\mathbb{R}}_{\lambda}^{n}$ be a (λ, μ) -strongly-compact set, then f(X) is also (λ, μ) -strongly-compact. In particular, it has a maximum and minimum on X.

Proposition (Intermediate value theorem)

Let $f : \mathbb{R}^{\mu}_{\lambda} \to \mathbb{R}^{\mu}_{\lambda}$ be (λ, μ) -strongly-continuous. Assume $f(a) \leq f(b)$. Then for all $y \in [\min(f(a), f(b)); \max(f(a), f(b))]$ there is $a \leq c \leq b$ such that f(c) = y.

Theorem (Extreme Value Theorem)

Let $f : \mathbb{R}^{\mu}_{\lambda} \to \mathbb{R}^{\mu}_{\lambda}$ be a (λ, μ) -strongly-continuous function. Let $X \subseteq \mathbb{R}^{\mu}_{\lambda}$ be a (λ, μ) -strongly-compact set, then f(X) is also (λ, μ) -strongly-compact. In particular, it has a maximum and a minimum on X.

Quentin Guilmant

fields.

Example

The following functions are strongly continuous :

- Polynomials, exponential, logarithm
- Arctan (inductive definition from Rubinstein-Salzedo [7])
- Composition of (λ, μ)-strongly-continuous functions

Surreal numbers Problem of integration Strongly continuous functions

11-26

2019-1

Example

The following functions are strongly continuous : • Polynomials, exponential, logarithm • Arctan (inductive definition from Rubinstein-Salzedo [7]) • Composition of (A, p)-strongly-continuous functions

Strongly continuous functions

Motivation : Analog

Perspectiv

Quentin Guilmant

Example

The following functions are strongly continuous :

- Polynomials, exponential, logarithm
- Arctan (inductive definition from Rubinstein-Salzedo [7])
- Composition of (λ, μ) -strongly-continuous functions

• $x \mapsto \begin{cases} x & x < \infty \\ x+1 & x > \infty \end{cases}$

• Composition

```
with surreal
numbers,
existing
methods
Cauchy completion
```

fields.

Computable Analysis over surreal fields

Problem of integration Motivation : Analog

Motivation : Analo computing

Some previous tries Handle the gaps, a

new notion of compacity.

Strongly continuous functions

Perspectives

Surreal numbers Problem of integration Strongly continuous functions

 $\begin{array}{l} \label{eq:complete} \\ \mbox{Ecomplet} \\ \mbox{Identify} & \mbox{Potential}, \mbox{ecommits}, \mbox{ec$

Quentin Guilmant

Example

The following functions are strongly continuous :

- Polynomials, exponential, logarithm
- Arctan (inductive definition from Rubinstein-Salzedo [7])
- Composition of (λ, μ) -strongly-continuous functions

• $x \mapsto \begin{cases} x & x < \infty \\ x+1 & x > \infty \end{cases}$

lssue

 $(\lambda,\mu)\text{-strongly-continuous functions are not stable under ring operations <math display="inline">+,\times$

Surreal numbers Problem of integration Strongly continuous functions

Example The following functions are strongly continuous : Polynomials, exponential, logarithm Arctan (inductive definition from Rubinstein-Salzedo [7]) Composition of (Λ_{i}) -strongly-continuous functions $\begin{cases} x & x \le \infty \end{cases}$

 $x \mapsto \begin{cases} x & x < \infty \\ x+1 & x > \infty \end{cases}$

 $(\lambda,\mu)\text{-strongly-continuous functions are not stable under ring operations <math display="inline">+,\times$

Strongly continuous functions Perspectives

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations

Sub-structure and Hahn series Gaps in the surreal

Computations with surreal numbers, existing methods Cauchy completion Computable Analysi

over surreal fields

integration

Motivation : Analog computing

Some previous tries

Handle the gaps, new notion of compacity.

Strongly continuou functions

Perspectives

Integration with Stone-Weierstrass

• Find a suitable ring of (λ, μ) -strongly-continuous functions

Surreal numbers Problem of integration Perspectives Integration with Stone-Weierstrass Integration with Stone-Weierstrass

+ Find a suitable ring of ($\lambda,\mu)$ -strongly-continuous functions

Quentin Guilmant

Perspectives

Integration with Stone-Weierstrass

• Find a suitable ring of (λ, μ) -strongly-continuous functions • On closed intervals, make them approachable by suitable strongly continuous functions for which we do know a

primitive (Stone-Weierstrass-like theorem)



 Find a suitable ring of (λ, μ)-strongly-continuous function · On closed intervals, make them anormarhable by suitable strongly continuous functions for which we do know a

primitive (Stone-Weierstrass-like theorem)

Integration with Stone-Weierstrass

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations

Hahn series Gaps in the surreal

Computations with surreal numbers, existing methods Cauchy completion

Computable Analysis over surreal fields

Integration Motivation : Analo computing Some previous trie Handle the gaps, a new notion of compacity.

Strongly continuou functions

Perspectives

Integration with Stone-Weierstrass



Integration with Stone-Weierstrass

Find a suitable ring of (λ, μ)-strongly-continuous function
 On closed intervals, make them approachable by suitable strongly continuous functions for which we do know a primitive (Stone-Weiserstrass-Bie theorem)
 Define integration with limits for such functions.

- Find a suitable ring of (λ, μ) -strongly-continuous functions
 - On closed intervals, make them approachable by suitable strongly continuous functions for which we do know a primitive (Stone-Weierstrass-like theorem)
 - Define integration with limits for such functions.

Quentin Guilmant

Introduction : Numbers

Surreal Numbers

Definitions and operations Sub-structure and Hahn series Gaps in the surreal fields.

Idea

Computations with surreal numbers, existing methods Cauchy completion Computable Analysis over surreal fields

Problem of integration

Motivation : Analog computing

Some previous trie

Handle the gaps, new notion of compacity.

Strongly continuou functions

Perspectives

Suitable restricted set theory

Gaps are "very" outside the surreal fields.

Surreal numbers Problem of integration Perspectives Suitable restricted set theory Suitable restricted set theory

Idth Gaps are "very" outside the surreal fields.

Quentin Guilmant

Perspectives

Suitable restricted set theory

Surreal numbers -26 Problem of integration H -Perspectives 2019-3 -Suitable restricted set theory Suitable restricted set theory

Gaps are "verv" outside the surreal fields

If we consider a suitable (with notions of constructivism) set theory (something like what is introduced in Barwise, [2]). strongly compact sets may be the actual compact sets and strongly continuous functions, the continuous function.

Idea

Gaps are "very" outside the surreal fields.

If we consider a suitable (with notions of constructivism) set theory (something like what is introduced in Barwise, [2]), strongly compact sets may be the actual compact sets and strongly continuous functions, the continuous function.
Surreal numbers

Quentin

Guilmant

Motivation : Analog

Norman L. Alling.

Foundations of analysis over surreal number fields., volume 141. Elsevier, Amsterdam, 1987.

Jon Barwise.

Admissible sets and structures, volume 7. Cambridge University Press, 2017.

J.H. Conway.

Ē

ē.

Ē.

On Numbers and Games. Ak Peters Series. Taylor & Francis, 2000.

Antongiulio Fornasiero

Integration on surreal numbers. Technical report, 2003.

Lorenzo Galeotti, Alexandru Baltag, Benno Van Den Berg, Yurii Khomskii, Löwe Benedikt, Drs, Arno Hugo Nobrega, Bernadette Bernie Pauly, and Rin. Computable analysis over the generalized baire space. 2015.

H. Gonshor and N.J. Hitchin.

An Introduction to the Theory of Surreal Numbers. London Mathematical Society Le. Cambridge University Press, 1986

Simon Rubinstein-Salzedo and Ashvin Swaminathan. Analysis on surreal numbers. arXiv preprint arXiv :1307.7392, 2013.

Lou van den Dries and Philip Ehrlich. Fields of surreal numbers and exponentiation. Fundamenta Mathematicae - FUND MATH, 167 :173-188, 01 2001.

Surreal numbers 9 Ň ÷. -Ó 201

D in terms 1 1N Commy On Numbers and Games Antongalia Parawina H. Conduct and N.J. Hitshin. As introduction in the Theory of Ramed Rambox. B Area Balance Joint of Aria Institut

Los on the Drin and Phily Delin.

Normal 1. Allog. Providence of analysis over annual number fields, volume 141.