Surreal Numbers, integration and computations

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LIX, École Polytechnique

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Surreal numbers

1 Introduction : Numbers
2 Surreal Numbers
Definitions and operations
Sub-structure and Hahn series
Gaps in the surreal fields.
3 Computations with surreal numbers, existing methods
Cauchy completion
Computable Analysis over surreal fields
4 Problem of integration
Motivation : Analog computing
Some previous tries
Handle the gaps, a new notion of compacity.
Strongly continuous functions
Perspectives

Outline

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1 Introduction : Numbers

Numbers

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- $\mathbb{R}$ is the unique field that is both Archimedean and complete. This properties and fundamental to prove fundamental theorems of analysis (TVE, TVI, Rolle, Mean Value Theorem, Fixed Point Theorems).
- With ordinal number instead of natural numbers we may get other things. We may get a lot of new numbers, that may look like the normal form of ordinal theorems.

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3 methods to think about surreal numbers

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No is the <u>class</u> of all surreal numbers. It contains $\mathbb{R}$, the ordinals and many (many (many)) other ones (<u>Ex</u> :$\frac{\omega}{2}$, $\sqrt{\omega}$, $\omega^{1/\omega} = \log \omega$).

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Surreal tree

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Surreal numbers
└Surreal Numbers
  └Definitions and operations
    └Surreal tree

Surreal tree

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Voir la figure

Surreal tree

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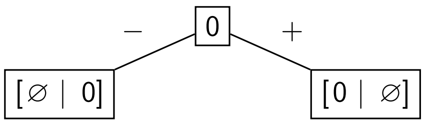
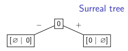
Surreal tree





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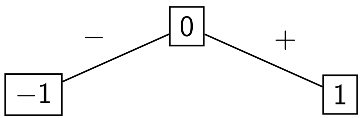

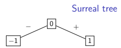

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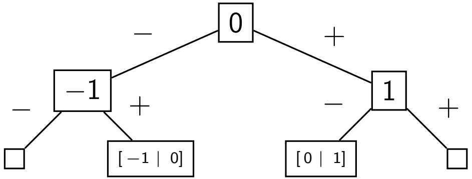

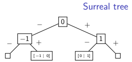

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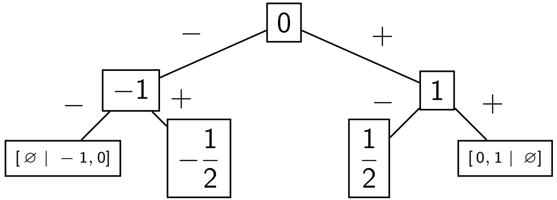

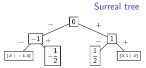

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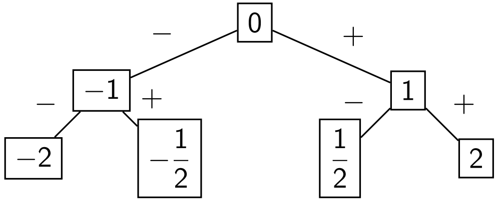

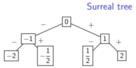

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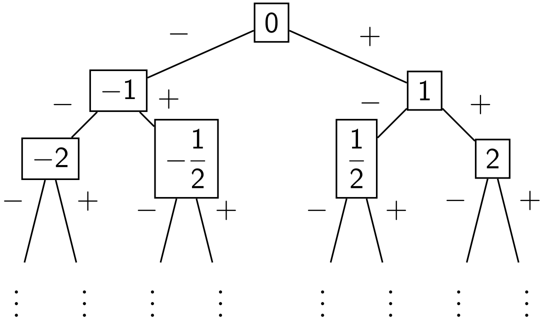

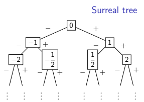

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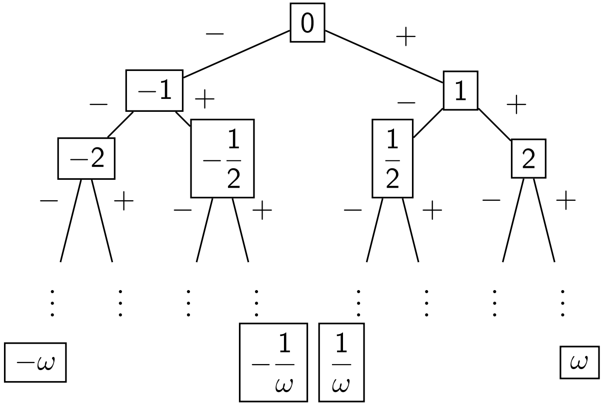

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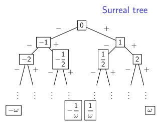

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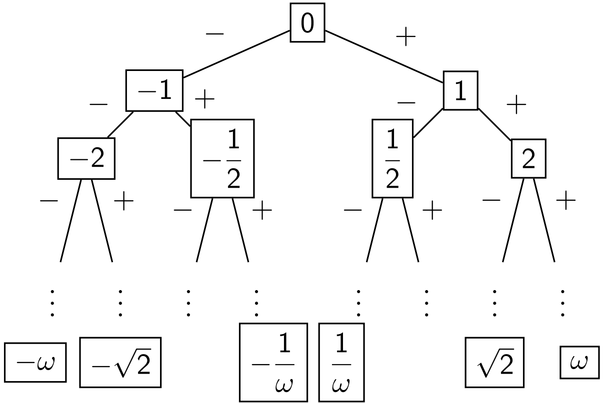

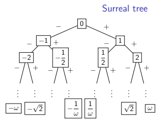

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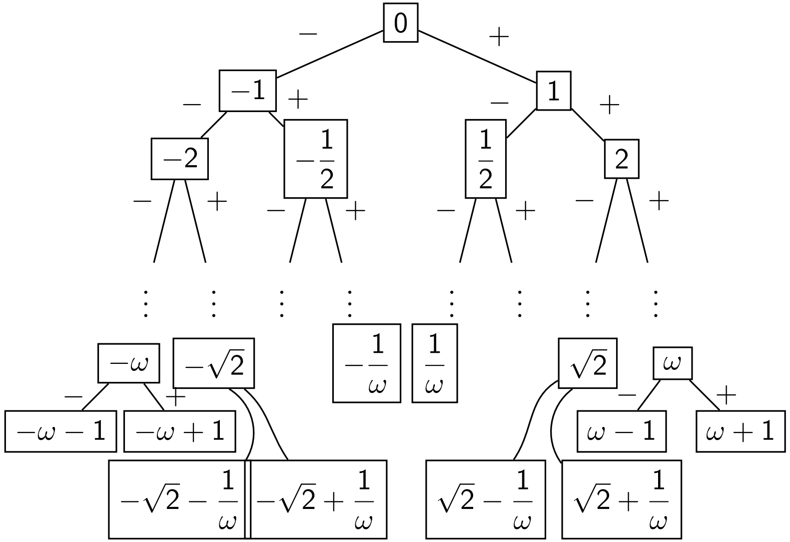

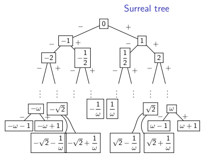

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Inductive definitions

Theorem (Conway)

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If $L < R$ are two *subsets* *of the surreal numbers then there is a unique x with minimum length such that $L < x < R$*

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Definition

$$x + y = [L_x + y,\ x + L_y \mid R_x + y,\ x + R_y]$$

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- *Addition has the uniformity property.*
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Surreal operations : Addition, example

$$x = \omega + \frac{3}{4} = \left[\mathbb{N}, \omega, \omega + \frac{1}{2} \,\middle|\, \omega + 1\right]$$
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$$\text{Then} \begin{cases} L_x + y = \{-3.5, -2.5, -1.5, -0.5, 0.5, \ldots\} \\ x + L_y = \{\omega - \frac{13}{4}\} \\ R_x + y = \{\omega - \frac{5}{2}\} \\ x + R_y = \{\omega + \frac{3}{4}, \omega - \frac{1}{4}, \omega - \frac{5}{4}, \omega - \frac{9}{4}\} \end{cases}$$

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Basically we have to choose between

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Simplicity property :

$$x + y = (+)^\omega - - - + - = \omega - \frac{11}{4}$$

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$$y = -\frac{7}{2} = [-4 \mid 0, -1, -2, -3]$$

Then $L_x + y = \{-3.5, -2.5, -1.5, -0.5, 0.5, \ldots\}$, $x + L_y = \{\omega - \frac{13}{4}\}$, $R_x + y = \{\omega - \frac{5}{2}\}$, $x + R_y = \{\omega + \frac{3}{4}, \omega - \frac{1}{4}, \omega - \frac{5}{4}, \omega - \frac{9}{4}\}$

$x + y = \left[\omega - \frac{13}{4} \mid \omega - \frac{5}{2}\right]$

$x + y = [(+)^\omega - - - - + + \mid (+)^\omega - - - +]$

Basically we have to choose between

$(+)^\omega - - - - + + +$ and $(+)^\omega - - - + -$

Simplicity property :

$x + y = (+)^\omega - - - + - = \omega - \frac{11}{4}$

Exemple de $+$

Surreal operations : Multiplication

Definition

$$x \times y = \left[\begin{array}{c} l_x y + x l_y - l_x l_y \\ r_x y + x r_y - r_x r_y \end{array} \middle| \begin{array}{c} l_x y + x r_y - l_x r_y \\ r_x y + x l_y - r_x l_y \end{array} \right]$$

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Proposition

- *Multiplication has the uniformity property*
- $(\textbf{No}_{<}, +, \times)$ *is field.*

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Surreal operations : Inverse

Proposition

Let $x = [L_x \mid R_x]$ in the canonical representation. Then $\frac{1}{x}$ is defined as follows.

- $\langle\rangle = 0$
- *For* $y_0, \ldots, y_n \in (L_x \cup R_x) \setminus \{0\}$,

$$\langle y_0, \ldots, y_n \rangle = \frac{1 - (x - y_n)\langle y_0, \ldots, y_{n-1}\rangle}{y_n}$$

$$\begin{cases} L_{1/x} = \{\langle y_0, \ldots, y_n \rangle \mid |\{i \mid y_i \in L_x\}| \in 2\mathbb{N}\} \\ R_{1/x} = \{\langle y_0, \ldots, y_n \rangle \mid |\{i \mid y_i \in L_x\}| \in 2\mathbb{N}+1\} \end{cases}$$

Then $$\frac{1}{x} = \left[L_{1/x} \mid R_{1/x}\right]$$

And the definition has the uniformity property.

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Proposition
Let $x = [L_x \mid R_x]$ in the canonical representation. Then $\frac{1}{x}$ is defined as follows.

Definition de l'inverse

Exponential function

Theorem (Gonshor [6])

It is possible to extend the function $\exp : \mathbb{R} \to \mathbb{R}$ *to* $\exp : \textbf{No} \to \textbf{No}$ *such that it has an inductive definition with the uniformity property.*

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There is a hierarchy of elementary extensions $\mathbb{R} \subseteq \textbf{No}_{<\lambda} \subseteq \textbf{No}$ *(for* λ *an* ε*-number) for the language of ordered fields together with* $\exp$ *(and restricted analytic functions).*

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Warning

sin and cos do not admit extension to surreal number. You would need to give sense to $\exp(\mathbf{i}\omega)$.

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- Even the exponential function has a nice definition ! In particular it is inductive with uniform property.
- The definition of the exponential function is very satisfying, it has all the suitable first order properties of the exponential function.
- What does it mean to orbit ω times around the origin ? Even worth : $\sqrt{\omega}$ times ?

Surreal numbers of bounded length

Notation

$$\textbf{No}_{<\alpha} = \{x \in \textbf{No} \mid \text{length}(x) < \alpha\}$$

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- So far we were interested in the whole class of surreal numbers. But there are sub-structures that may be interesteing. Typically, what about the set of surreal numbers of bounded length (or birthday).
- The very first question is, what are the conditions on α so that we have usual algebraic structures.

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Proposition (Van den Dries and Ehrlich [8], corollaries 3.1, 4.4 and 4.9)

$\mathbf{No}_{<\lambda}$ is

- *an additive group iff λ is additive (i.e has form $\lambda = \omega^{\alpha}$ for some ordinal α)*
- *is a ring iff λ is multiplicative (i.e has form $\lambda = \omega^{\omega^{\alpha}}$ for some ordinal α)*
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- The very first question is, what are the conditions on α so that we have usual algebraic structures.
- The answer is very intuitive : it is basically the desired property applied to the ordinal that bounds the length.

Interlude : Hahn series

Definition (Hahn series field)

Let $\mathbb{K}$ be a field and Γ an Abelian ordered group. The associated formal power series field denoted $\mathbb{K}((t^\Gamma))$ is

$$\left\{ \sum_{\gamma\in\Gamma} r_\gamma t^\gamma \;\middle|\; r_\gamma \in \mathbb{K},\ \operatorname{supp}(x) := \{\gamma \mid r_\gamma \neq 0\} \text{ is well ordered} \right\}$$

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*Every x in **No** can be written in a unique way as $x = \sum_{i<\alpha} r_i \omega^{a_i}$ with $r_i \in \mathbb{R}$ and the $a_i \in$ **No** decreasing and **No** $\simeq \mathbb{R}((t^{\mathbf{No}}))$ (ordered fields isomorphism).*

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So far we have seen the the two first definition of the surreal numbers. What about the third one ?

- We first define Hahn series.
- This is an ordered field
- We have a very good isomorphism that links the sign expansions and the Hahn series. Moreover the the sign expansion can be "easily" deduced from the Hahn series. Finally the Hahn series of ordinal number seen as surreal numbers are their normal forms as usual ordinals.

Sub-fields from Hahn series

Theorem (Alling [1], Van den Dries, Ehrlich [8])

Let $\mathbb{K}$ be real-closed and Γ divisible. Let λ be an ε-number. Then

$$\mathbb{K}_\lambda^\Gamma := \left\{ x \in \mathbb{K}((t^\Gamma)) \;\middle|\; \operatorname{supp} x \textit{ has order type lower than } \lambda \right\}$$

is a real-closed field.

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Cuts and gaps

Definition

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Types of Gaps in **No**

Proposition (Conway, [3])

*Gaps in **No** may be of the form :*

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Les deux types de gap

Please notice that there are only non trivial gaps. That is because if there were a non-trivial gap, one of the set L or R involved in the gap would actually be a proper class.

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The Cauchy-completion of $\mathbb{R}^{\mu}_{\lambda}$ *is*

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- Def : Coinitial, Cofinal, Coinitiality, Cofinality.
- With Hahn series, we have a nice Cauchy completion
- The remaining gaps are exactly what we expect. In particular, in type 1 gaps we just get rid of the case where the (a_i)s are coinitial.

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Dedekind completion would introduce new elements that are even not surreal number (such as ∞). Of course you may say there are surreal numbers that are not the considered field and that would be able to play the role of ∞. But adding them create even more gaps because you want to get a field at the end !

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Intervals representation (1/2)

- $\mathbb{R}_\lambda^\mu$ admits a notation over $\{0,1\}^{<\lambda}$. It is dense in $\widetilde{\mathbb{R}_\lambda^\mu}$.
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- Give $\widetilde{\mathbb{R}_\lambda^\mu}$ a structure of an effective (topological) space.

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Interlude : Effective space

Definition (λ-effective space)

A λ-effective space is a triplet $\mathcal{M} = (M, \sigma, \nu)$ with M a set, $\sigma \subseteq 2^M$ a collection of subsets of M such that

$$x = y \quad \Leftrightarrow \quad \{A \in \sigma \mid x \in A\} = \{A \in \sigma \mid y \in A\}$$

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Outline

4 Problem of integration

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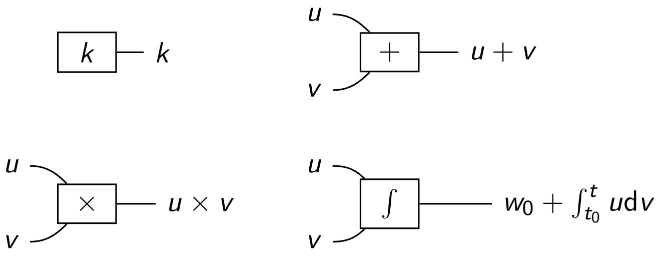
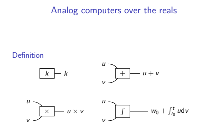
Analog computers over the reals

Definition





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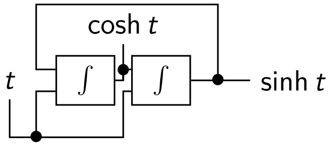

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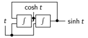

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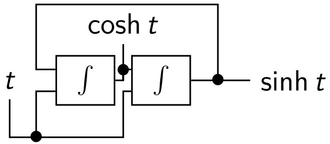

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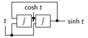

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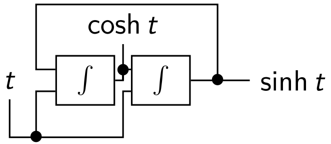

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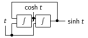

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A solution exists

The solution is unique

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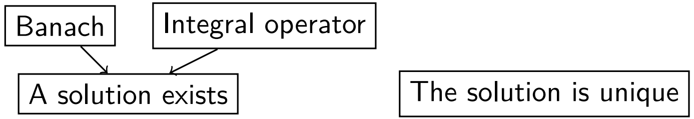

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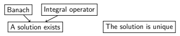

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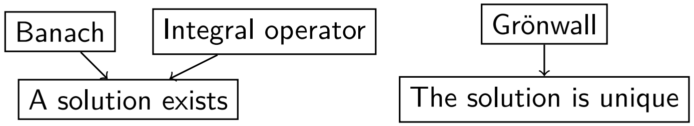

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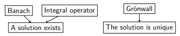

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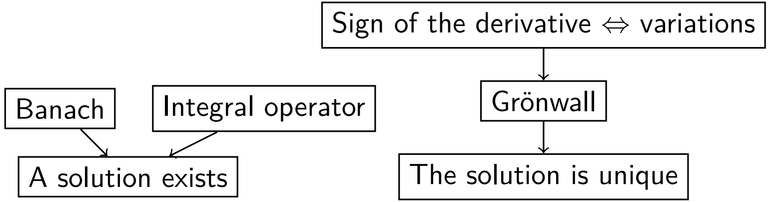

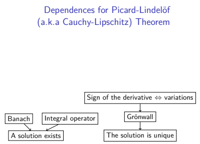

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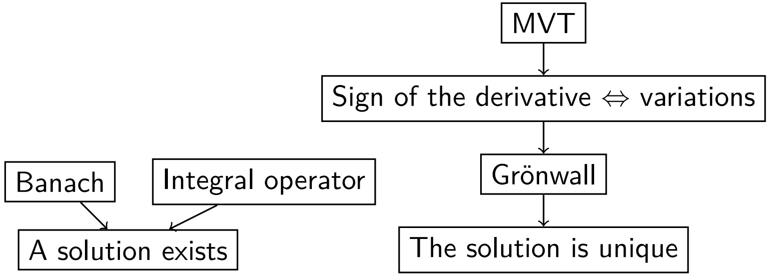

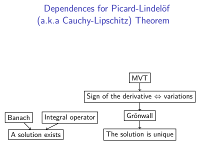

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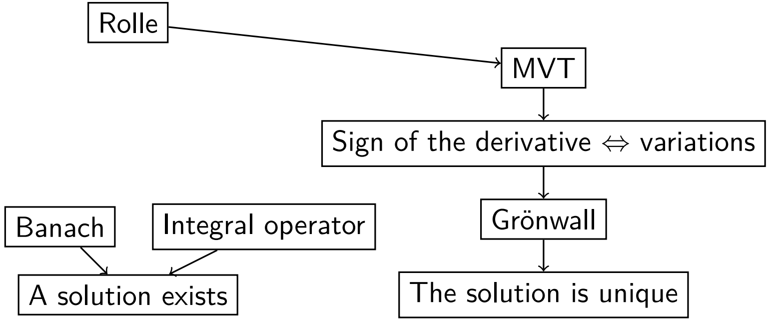

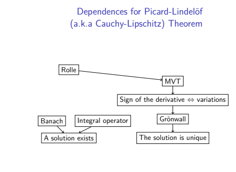

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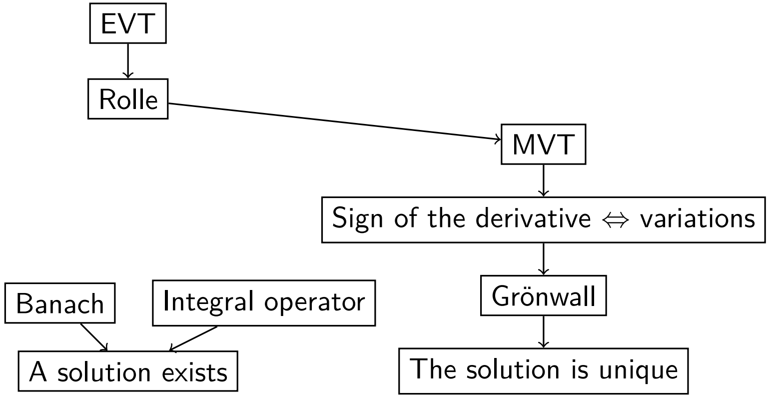

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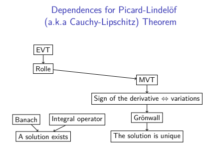

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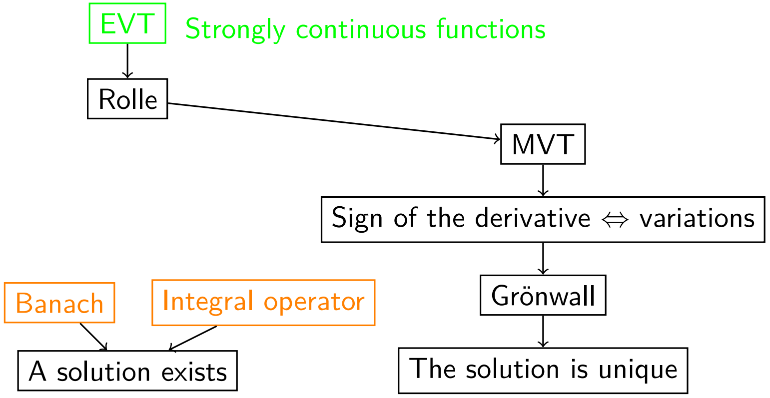

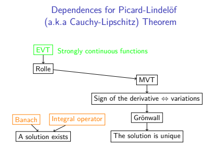

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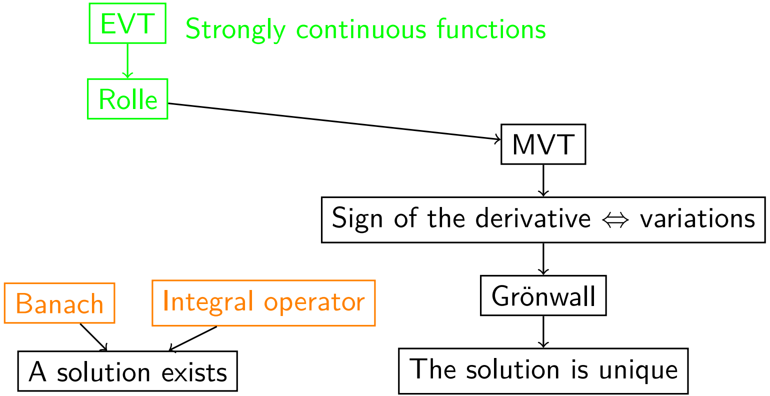

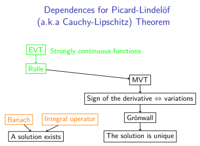

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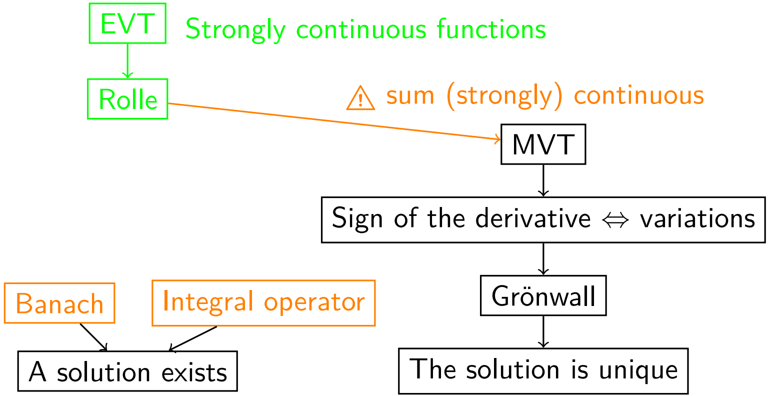

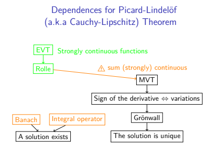

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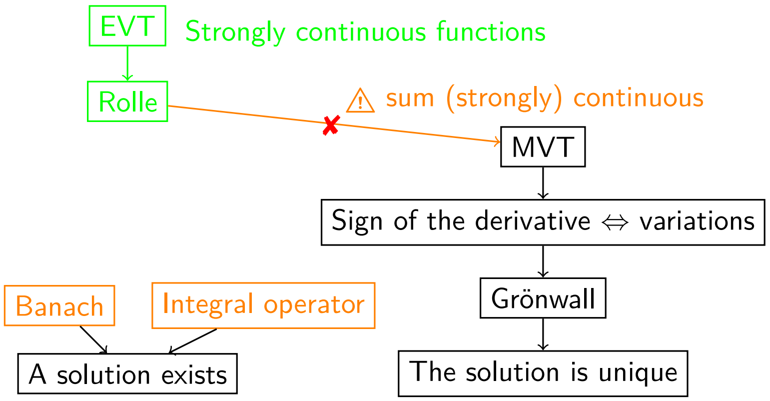

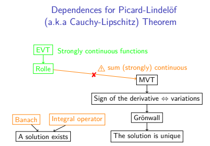

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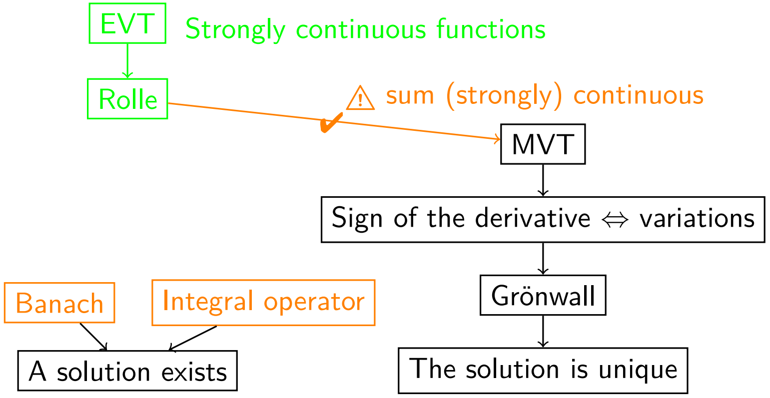

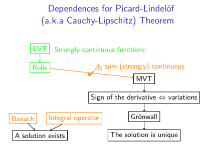

Generalized Riemann Sums

Definition (Rubinstein-Salzedo, Swaminathan [7])

Let $\mathbb{K} \subseteq \mathbf{No}$, $\mathcal{F} \subseteq \bigcup_n (\mathbb{K}^n \to \mathbb{K})$ and $\mathscr{F}$ be its $(+, \times, \circ)$-closure.

Let $f : [a; b] \to \mathbb{K}$ continuous. If there is $g \in \mathscr{F}$ such that

$$\forall n \in \mathbb{N}\ \forall a \le c \le d \le b \quad g(n, c, d) = \sum_{i=0}^{n} \frac{d-c}{n} f(c + i\frac{d-c}{n})$$

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TFA	✔(not unique)	✔(not unique)
Rolle	✔	✔ (strongly continuous)
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- *X is closed and for any $Y \subseteq X$ such that $\sup Y$ (resp. $\inf Y$) is a gap, there is $Z \subseteq X$ such that $\inf Z = \sup Y$ (resp. $\sup Z = \inf Y$)*

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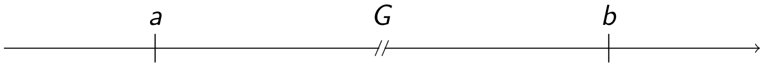
An example



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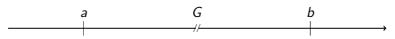

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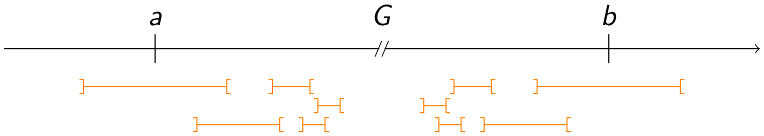

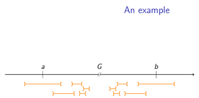

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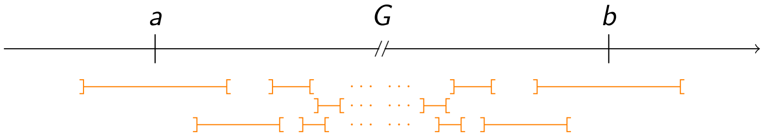

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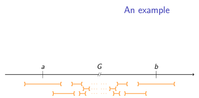
Surreal numbers
└Problem of integration
└Handle the gaps, a new notion of compacity.
└An example



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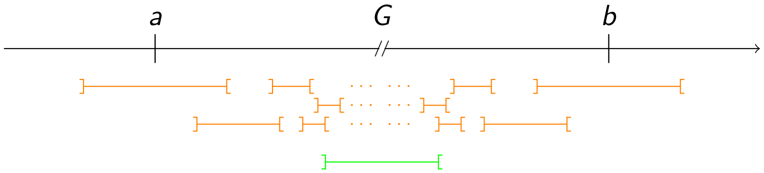

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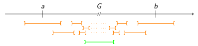

Strongly continuous functions

Definition

A function $f : \widetilde{\mathbb{R}_\lambda^\mu} \to \widetilde{\mathbb{R}_\lambda^\mu}$ is said to be (λ, μ)-strongly-continuous if it is continuous and for any non-trivial gap $G = [L \mid R]$ of $\widetilde{\mathbb{R}_\lambda^\mu}$ either f has a limit in G that is reached on any neighborhood of G or there is a non-trivial gap $H = [A \mid B]$ such that for any neighborhood J of H there is a neighborhood I of G such that

$$x \in I \implies f(x) \in J$$

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Basic analysis of strongly continuous functions

Proposition (Intermediate value theorem)

Let $f : \widetilde{\mathbb{R}}_\lambda^\mu \to \widetilde{\mathbb{R}}_\lambda^\mu$ be (λ,μ)-strongly-continuous . Assume $f(a) \leq f(b)$. Then for all $y \in [\min(f(a), f(b))\,;\max(f(a), f(b))]$ there is $a \leq c \leq b$ such that $f(c) = y$.

Theorem (Extreme Value Theorem)

Let $f : \widetilde{\mathbb{R}}_\lambda^\mu \to \widetilde{\mathbb{R}}_\lambda^\mu$ be a (λ,μ)-strongly-continuous function. Let $X \subseteq \widetilde{\mathbb{R}}_\lambda^\mu$ be a (λ,μ)-strongly-compact set, then $f(X)$ is also (λ,μ)-strongly-compact . In particular, it has a maximum and a minimum on X.

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