Lowness of the piegeonhole principle

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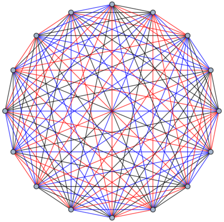


Section 1

Ramsey Theory

Motivation

It all started with this guy...

Theorem (Ramsey's theorem)

Let $n \geqslant 1$. For each coloration of $[\omega]^n$ in a finite number of color, there exists a set $X \in [\omega]^\omega$ such that each element of $[X]^n$ has the same color (X is said to be monochromatic).

Motivation

Ramsey Theory

A general question

Suppose we have some mathematical structure that is then cut into finitely many pieces. How big must the original structure be in order to ensure that at least one of the pieces has a given interesting property ?

Examples :

- Van der Waerden's theorem
- Hindman's theorem
- ...

Motivation

Example (Van der Waerden's theorem)

For any given c and n, there is a number $w(c,n)$, such that if $w(c,n)$ consecutive numbers are colored with c different colors, then it must contain an arithmetic progression of length n whose elements all have the same color.

We know that :

$$w(c,n) \leqslant 2^{2^{c^{2^{2^{n+9}}}}}$$

Example (Hindmam's theorem)

If we color the natural numbers with finitely many colors, there must exists a monochromatic infinite set closed by finite sums.

Partition regularity

Theorems in Ramsey theory often assert, in their stronger form, that certain classes are *partition regular* :

Definition (Partition regularity)

A *partition regular* class is a non-empty collection of sets $\mathcal{L} \subseteq 2^\omega$ such that :

- $\mathcal{L}$ is upward closed : If $X \in \mathcal{L}$ and $X \subseteq Y$, then $Y \in \mathcal{L}$
- If $X \in \mathcal{L}$ and $Y_0 \cup \cdots \cup Y_k \supseteq X$, then there is $i \leqslant k$ such that $Y_i \in \mathcal{L}$

Proper partition regular classes are exactly the complements of proper set theoretic ideals :

Definition (Ideals)

An *ideal* class is a non-empty collection of sets $\mathcal{I} \subseteq 2^\omega$ such that :

- $\mathcal{I}$ is downward closed : If $X \in \mathcal{L}$ and $X \supseteq Y$, then $Y \in \mathcal{I}$
- If $Y_0, \ldots, Y_k \in \mathcal{I}$, then $Y_0 \cup \cdots \cup Y_k \in \mathcal{I}$

Partition regularity

The following classes are partition regular :

Classical combinatorial results :

- The class of infinite sets
- The class of sets with positive upper density
- The class of sets X s.t. $\sum_{n \in X} \frac{1}{n} = \infty$
- The class of sets containing arbitrarily long arithmetic progressions (Van der Waerden's theorem)
- The class of sets containing an infinite set closed by finite sum (Hindman's theorem)

... and *new* type of results involving computability :

- Given X non-computable, the class of sets containing an infinite set which does not compute X (Dzhafarov and Jockusch)

Seetapun's theorem

Theorem (Dzhafarov and Jockusch)

Given X non-computable, Given $A^0 \cup A^1 = \omega$, there exists $G \in [A^0]^\omega \cup [A^1]^\omega$ such that G does not compute X.

This theorem comes from Reverse mathematics :

What is the computational strength of Ramsey's theorem ?

that is, given a computable coloring of say $[\omega]^2$, must all monochromatic sets have a specific computational power ?

Theorem (Seetapun)

For any non-computable set X and any computable coloring of $[\omega]^2$, there is an infinite monochromatic set which does not compute X.

Theorem (Jockusch)

There exists a computable coloring of $[\omega]^3$, every solution of which computes $\emptyset'$.

Modern approach of Seetapun's theorem

Modern approach of Seetapun's theorem (Cholak, Jockusch, Slaman) :

Definition

A set C is $\{R_n\}_{n\in\omega}$-*cohesive* if $C \subseteq^* R_n$ or $C \subseteq^* \overline{R_n}$ for every n.

Definition

A coloring $c : \omega^2 \to \{0,1\}$ is *stable* if $\forall x \ \lim_{y\in\omega} c(x,y)$ exists.

- Given a computable coloring $c : \omega^2 \to \{0,1\}$, let $R_n = \{y \ : \ c(n,y) = 0\}$. Let C be $\{R_n\}_{n\in\omega}$-cohesive. Then c restricted to C is stable.
- Let c be a stable coloring. Let A_c be the $\underline{\Delta^0_2(c)}$ set defined as $A_c(x) = \lim_y c(x,y)$. An infinite subset of A_c or of $\overline{A_c}$ can be used to compute a solution to c.

$\to$ Find a cohesive set C (cohesive for the recursive sets) which does not compute X and use Dzhafarov and Jockusch relative to C with $A_{c\restriction C}$.

The general question

The following version of Dzhafarov and Jockusch's is also true :

Theorem (Dzhafarov and Jockusch)

Let X be non-computable. The class of sets

$$\{A \; : \; \text{There exists } G \in [A]^\omega \text{ such that } G \text{ does not compute } X\}$$

is partition regular.

Dzhafarov and Jockusch's theorem is sometimes called strong cone avoidance of RT^1_2 : the instance of RT^1_2 we consider does not need to be computable. We study here the folllowing general question, that we derive from Dzhafarov and Jockusch's

What computational power can we encode inside every infinite subsets of both two halves of ω ?

Section 2

Splitting ω in two

The question

What can we encode inside every infinite subsets of both two halves of ω ?

A splitting :

Such that :

- **Each infinite subset** of the blue part has some comp. power
- **Each infinite subset** of the red part has some comp. power

Answer : Not much...

A precision

What if we drop the complement thing ?

Consider any set X. Then we can encode X into every infinite subset of a set A the following way : We let A be all the integers which correspond to an encoding of the prefixes of X (using some computable bijection between 2^ω and ω).

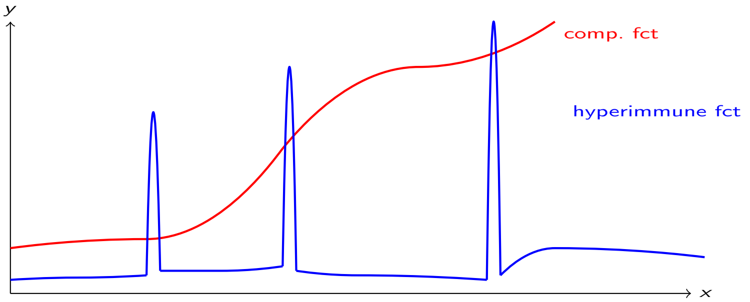
$$\sigma_0 \prec \sigma_1 \prec \sigma_2 \prec \dots X$$

$$A(n) = 1 \text{ iff } n \text{ encodes } \sigma_n \text{ for some } n$$

Encoding Hyperimmunity

Definition (Hyperimmunity)

A set X is of *hyperimmune degree* if X computes a function $f : \omega \to \omega$, which is not dominated by any computable function.



Theorem

There exists a covering $A^0 \cup A^1 \supseteq \omega$, such that every $X \in [A^0]^\omega \cup [A^1]^\omega$ is of hyperimmune degree.

Encoding Hyperimmunity

Theorem

There exists a covering $A^0 \cup A^1 \supseteq \omega$, such that every $X \in [A^0]^\omega \cup [A^1]^\omega$ is of hyperimmune degree.

We split ω by alternating larger and larger blocks of consecutive integers in A^0 and A^1.

For X infinite subset of A^0 or A^1, the hyperimmune function is given by $f(n)$ to be the n-th number which appears in X.

Encoding DNC

Definition (Diagonally non-computable degree)

A set X is of *DNC degree* (diagonally non-computable) if X computes a function $f : \omega \to \omega$, such that $f(n) \neq \Phi_n(n)$ for every n.

Theorem

The following are equivalent for a set X :

- *X is of DNC degree.*
- *X computes a function which on input n can output a string of Kolmogorov complexity greater than n.*
- *X computes an infinite subset of a Martin-Löf random set.*

Encoding DNC

Definition (Informal definition of Kolmorogov complexity)

We say $K(\sigma) \geqslant n$ if the size of the smallest program which outputs σ is at least n.

Definition (Informal definition of Martin Löf randomness)

We say X is Martin Löf random if the Kolmogorov complexity of each of its prefix σ is greater than $|\sigma|$.

Theorem

X is of DNC degree iff X computes an infinite subset of a Martin-Löf random set.

$$\begin{array}{ll} & 001011101010011011001101001011010110010101010\ldots \\ \to & 000010000000001000000000000001000110000000010\ldots \\ \to & 111111111011111111011111101111111110111101111\ldots \end{array}$$

Encoding enumerating non-enumerable things

Theorem [Tennenbaum, Denisov]

There exists a computable order of ω, of order type $\omega + \omega^*$ which has no infinite ascending or descending c.e. sequence.

Consider $A \subseteq \omega$ the initial segment of order-type ω.

- Any infinite subset $X \subseteq A$ enumerates A (by enumerating things smaller than elements of X)
- Any infinite subset of $X \subseteq \overline{A}$ enumerates $\overline{A}$ (by enumerating things larger than elements of X)

Corollary [Tennenbaum, Denisov]

There exists a set A such that every set $G \in [A]^\omega \cup [\overline{A}]^\omega$ can make c.e. something which is not c.e.

Cone avoidance

Theorem [Dzhafarov and Jockusch]

Let $X \subseteq \omega$ be non-computable. For every covering $A^0 \cup A^1 \supseteq \omega$, we have some $G \in [A^0]^\omega \cup [A^1]^\omega$ such that $G \not\geq_T X$.

The proof uses computable Mathias Forcing : Dzhafarov and Jockusch's technic have then been enhanced an reused in various manner by multiple authors to show other results of the same type, that we shall now expose.

Theorem [Strong form of Dzhafarov and Jockusch]

Let $X \subseteq \omega$ be non-computable. The class of sets

$$\{A \ : \ \text{There exists } G \in [A]^\omega \text{ such that } G \text{ does not compute } X\}$$

is partition regular.

More on cone avoidance

Theorem [Dzhafarov and Jockusch]

Let $X \subseteq \omega$ be non-c.e. The class of sets

$$\{A \ : \ \text{There exists } G \in [A]^\omega \text{ such that } X \text{ is not c.e. in } G\}$$

is partition regular

But we cannot avoid more than one c.e. set. On the other hand :

Theorem [Dzhafarov and Jockusch]

Let $\{X_n\}_{n \in \omega}$ be all non-computable. The class of sets

$$\{A \ : \ \text{There exists } G \in [A]^\omega \text{ such that } G \text{ computes no } X_n\}$$

is partition regular

PA degrees

Definition

A set X is of P.A. degree if X computes a complete and consistent extension of Peano arithmetic.

Theorem

The following are equivalent :

- *X is of P.A. degree.*
- *X is diagonally non-computable with a $\{0,1\}$-valued function.*
- *X computes an infinite path in any non-empty Π^0_1 class.*

Theorem (Liu)

The class of sets

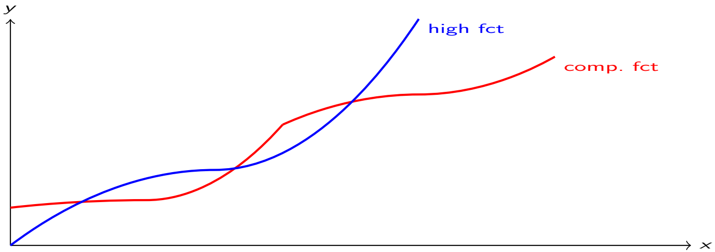
$$\{A \ : \ \text{There exists } G \in [A]^\omega \text{ which is not of PA degree } \}$$

is partition regular

Non high

Definition

A set X is high if it computes a function which eventually grows faster than any computable function.



Theorem (M., Patey)

The class of sets

$$\{A \ : \ \text{There exists } G \in [A]^{\omega} \text{ such that } G \text{ is not high } \}$$

is partition regular

Non high

Theorem (Martin)

The following are equivalent for a set X :

- *X is high*
- $X' \geqslant_T \emptyset''$

Theorem (M., Patey)

Let $X \subseteq \omega$ be non-$\emptyset'$-computable. The class of sets

$$\{A \ : \ \text{There exists } G \in [A]^\omega \text{ such that } G' \text{ does not compute } X\}$$

is partition regular

The proof uses of new forcing technique that builds upon Mathias forcing to control the second jump.

Partition regularity is in particular a key concept of the used forcing.

Computing random sets

Theorem (Liu)

Let f be a computabe order function. The class of sets

$$\{A \ : \ \text{There exists } G \in [A]^\omega \text{ which is not of } DNC_f \text{ degree }\}$$

is partition regular

Fact

Every Martin-Löf random Z is $DNC_{n \mapsto 2^n}$, that is, Z computes a DNC function bounded by $n \mapsto 2^n$.

Corollary [Liu]

The class of sets

$$\{A \ : \ \text{There exists } G \in [A]^\omega \text{ which compute no MLR set }\}$$

is partition regular

Computing generic sets

Definition

A set is *weakly-n-generic* if it is in every $\Sigma^0_1(\emptyset^{(n-1)})$ dense open set. It is 1-*generic* if for every $\Sigma^0_1(\emptyset^{(n-1)})$ open set U, it is in U or in the interior of the complement of U.

Theorem

There exists a covering $A^0 \cup A^1 \supseteq \omega$, such that for every $G \in [A^0]^\omega \cup [A^1]^\omega$ we have that G computes a 2-generic.

This is because any function which is not bounded by any Δ^0_3 function can compute a 2-generic. This does not work anymore with weakly-3-genericity and above.

Computing generic sets

Theorem (Andrews, Gerdes, Miller)

There is a function bounded by no Δ^0_3 function which computes no weakly-3-generic set.

The previous theorem gives us material for the following conjecture :

Conjecture

The class of sets

$\{A \ : \ \text{There is } G \in [A]^\omega \text{ which computes no weakly-3-generic set } \}$

is partition regular

Iterating throught the ordinals

Theorem (M., Patey)

Let $\alpha < \omega_1^{ck}$. Let X be non $\emptyset^{(\alpha)}$-computable. The class of sets

$$\{A \ : \ \text{there is } G \in [A]^\omega \text{ such that } X \text{ is not } G^{(\alpha)}\text{-computable}\}$$

is partition regular

Theorem (M., Patey)

Let X be non Δ_1^1. The class of sets

$$\{A \ : \ \text{there is } G \in [A]^\omega \text{ such that } X \text{ is not } \Delta_1^1(G)\}$$

is partition regular

Theorem (M., Patey)

The class of sets

$$\{A \ : \ \text{there is } G \in [A]^\omega \text{ such that } \omega_1^X = \omega_1^{ck}\}$$

is partition regular

Computing cohesive sets

Definition (Cohesiveness)

A set X if p-cohesive if for any primitive recursive set R_e we have $X \subseteq^* R_e$ or $X \subseteq^* \overline{R_e}$

Theorem (Folklore)

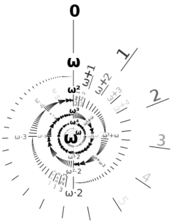
A set X computes a p-cohesive set iff X' is $\mathrm{PA}(\emptyset')$, that is, iff X' computes a function $f : \omega \to \{0,1\}$ such that $f(n) \neq \Phi_e^{\emptyset'}(e)$.

Theorem (M., Patey)

For every Δ^0_2 set A, there is an element $G \in [A]^\omega \cup [\overline{A}]^\omega$ such that G' is not $\mathrm{PA}(\emptyset')$.

Question

Is the former true for any set A?



Section 3

Iterating through the ordinals

The goal

Theorem (M., Patey)

Let $X \subseteq \omega$ be non-$\emptyset'$-computable. The class of sets

$$\{A \;:\; \text{There exists } G \in [A]^\omega \text{ such that } G' \text{ does not compute } X\}$$

is partition regular

Does this generalizes to any jump ?

Fact

The first "second jump control" forcing did not generalize to the third jump control.

Largeness and partition regularity

Definition (Largeness)

A *largeness* class is a collection of sets $\mathcal{L} \subseteq 2^\omega$ such that :

- $\mathcal{L}$ is upward closed : If $X \in \mathcal{L}$ and $X \subseteq Y$, then $Y \in \mathcal{L}$
- If $Y_0 \cup \dots \cup Y_k \supseteq \omega$, then there is $i \leqslant k$ such that $Y_i \in \mathcal{L}$
- If $X \in \mathcal{L}$ then $|X| \geqslant 2$

Definition (Partition regularity)

A *partition regular* class is a collection of sets $\mathcal{L} \subseteq 2^\omega$ such that :

- $\mathcal{L}$ is a largeness class
- If $X \in \mathcal{L}$ and $Y_0 \cup \dots \cup Y_k \supseteq X$, then there is $i \leqslant k$ such that $Y_i \in \mathcal{L}$

We add the condition $|X| \geqslant 2$ to ensure that $\mathcal{L}$ contains only infinite elements.

Generalities

Proposition

A partition regular class $\mathcal{L}$ contains only infinite sets.

Proposition

Let $\mathcal{L}$ be a partition regular class. Then $\mathcal{L}$ is closed by finite change of its elements. Furthermore if $\mathcal{L}$ is measurable it has measure 1.

Proof sketch :

$\mathcal{L}$ contains only infinite set

$\rightarrow$ $\mathcal{L}$ is closed by finite change

$\rightarrow$ $\mathcal{L}$ has measure 0 or 1

$\rightarrow$ If $\mathcal{L}$ has measure 0, sufficiently MLR Z and $\omega - Z$ are not in $\mathcal{L}$

$\rightarrow$ But Z or $\omega - Z$ must be in $\mathcal{L}$. Contradiction.

$\rightarrow$ $\mathcal{L}$ has measure 1

Generalities

Proposition (Compactness for largeness classes)

Suppose $\{\mathcal{A}_n\}_{n\in\omega}$ is a collection of largeness classes with $\mathcal{A}_{n+1} \subseteq \mathcal{A}_n$. Thus $\bigcap_{n\in\omega} \mathcal{A}_n$ is a largeness class.

Proposition (Compactness for partition regular classes)

Suppose $\{\mathcal{L}_n\}_{n\in\omega}$ is a collection of partition regular classes with $\mathcal{L}_{n+1} \subseteq \mathcal{L}_n$. Thus $\bigcap_{n\in\omega} \mathcal{L}_n$ is partition regular.

Proposition

Let $\mathcal{A}$ be any set. Then $\mathcal{A}$ is a largeness class iff the set

$$\mathcal{L}(\mathcal{A}) = \{X \in 2^\omega \ : \ \forall k \ \forall X_0 \cup \cdots \cup X_k \supseteq X \ \exists i \leqslant k \ X_i \in \mathcal{A}\}$$

is a partition regular subclass of $\mathcal{A}$ (in which case it is the largest).

Π_2^0 partition regular classes

Proposition

If $\mathcal{U}$ is a Σ_1^0 large class. Then $\mathcal{L}(\mathcal{U})$ is a Π_2^0 partition regular class.

Proposition

If $\mathcal{U}$ is a Σ_1^0 upward closed class. Then predicate

$$\mathcal{U} \text{ is large}$$

is Π_2^0.

Fix k, the class of element :

$$\{Y_0 \oplus \cdots \oplus Y_k \ : \ X \subseteq Y_0 \oplus \cdots \oplus Y_k \wedge \forall i < k \ Y_i \notin \mathcal{U}\}$$

is a $\Pi_1^0(X)$ class uniformly in X.

A glance at the forcing idea

Let (σ, X) be a condition.

- $\sigma \mathrel{?}\!\vdash \exists n\ \Phi(G, n)$ iff
$$\{Y\ :\ \exists n\ \exists \tau \subseteq Y\ \Phi(\sigma \cup \tau, n)\} \text{ is a largeness class}$$
- $\sigma \mathrel{?}\!\vdash \exists n\ \forall m_0 \dots Q m_k \Phi(G, n, m_0, \dots, m_k)$ iff
$$\{Y\ :\ \exists n\ \exists \tau \subseteq Y\ \sigma \cup \tau \mathrel{?}\!\nvdash \exists m_0 \dots \neg Q m_k \neg \Phi(G, n, m_0, \dots, m_k)\}$$
is a largeness class

→ If yes, X is in the largeness class. Take an extention of $\tau \geq \sigma$ with $\tau \subseteq X$

→ If no, there is a cover $Y_0 \cup \dots \cup Y_k \supseteq \omega$, such that for every extention $\tau \geq \sigma$ in Y_i and every n, “something is satisfied”. Take an extention $Y_i \cap X \subseteq X$ for the “right” Y_i.

Canonical Π^0_2 partition regular classes

The following classes are Π^0_2 partition regular classes.

Exemple

For X c.e. :

$$\mathcal{L}_X = \{Y \ : \ |X \cap Y| = \infty\}$$

Exemple

The class :

$$\mathcal{L}_{1/n} = \left\{X \ : \ \sum_{n \in X} 1/(1+n) = \infty\right\}$$

Exemple

The class :

$$\mathcal{L}_W = \{X \ : \ X \text{ contains arbitrarily long arithmetic progressions}\}$$

Minimal largeness classes

The challenge is to **fix in advance** all the possible largeness classes we want to work with, whithout being definitionally too complex.

Notation

For $C \subseteq \omega$ we write $\mathcal{U}_C = \bigcap_{e \in C} \mathcal{U}_e$

Definition (M., Patey)

Let $\mathcal{M}$ be a countable set. A largeness class $\mathcal{U}_C$ is $\mathcal{M}$-minimal if for every $\Sigma^0_1(X)$ class $\mathcal{U}$ for $X \in \mathcal{M}$ we have :

- $\mathcal{U}_C \subseteq \mathcal{U}$
- or $\mathcal{U} \cap \mathcal{U}_C$ is not a largeness class

Cohesive largeness classes

Definition (M., Patey)

Let $\mathcal{M}$ be a countable set. A largeness class $\mathcal{L}$ is $\mathcal{M}$-cohesive if for every $X \in \mathcal{M}$ we have :

- $\mathcal{L} \subseteq \mathcal{L}_X$
- or $\mathcal{L} \subseteq \mathcal{L}_{\overline{X}}$

Proposition (M., Patey)

Let $\mathcal{M}$ be a Scott set. An $\mathcal{M}$-cohesive largeness class contains a unique $\mathcal{M}$-minimal largeness class.

Notation

Let $\mathcal{M}$ be a Scott set and $\mathcal{U}_C$ be an $\mathcal{M}$-cohesive class. Then $\langle \mathcal{U}_C \rangle$ is the unique minimal largness subclass of $\mathcal{U}_C$.

The forcing (1)

Let $\{M_\alpha\}_{\alpha<\omega_1^{ck}}$ be such that

- M_α codes for a countable Scott set $\mathcal{M}_\alpha$
- $\emptyset^{(\alpha)}$ is uniformly coded by an element of $\mathcal{M}_\alpha$
- Each M'_α is uniformly computable in $\emptyset^{(\alpha+1)}$

Let $\{C_\alpha\}_{\alpha<\omega_1^{ck}}$ be such that :

- $\mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$ is an $\mathcal{M}_\alpha$-cohesive largeness class
- $\beta < \alpha$ implies $\mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha} \subseteq \langle \mathcal{U}_{C_\beta}^{\mathcal{M}_\beta} \rangle$
- Each C_α is coded by an element of $\mathcal{M}_{\alpha+1}$ uniformly in α and $M_{\alpha+1}$.

Let $\mathcal{S} = \bigcap_{\alpha<\omega_1^{ck}} \mathcal{U}_{C_\alpha}^{\mathcal{M}_\alpha}$. At least one among A or $\overline{A}$ belongs to $\mathcal{S}$

The forcing (2)

Let A be such that $A \in \mathcal{S}$. Forcing conditions are Mathias conditions (σ, X) such that :

- $\sigma \subseteq A$
- $X \subseteq A$
- $X \cap \{0, \dots, |\sigma|\} = \varnothing$
- $X \in \mathcal{S}$

Theorem (M., Patey)

Let B be not $\Delta^0_1(\emptyset^{(\alpha)})$ for $\alpha < \omega_1^{ck}$. If G is sufficiently generic then B is not $\Delta^0_1(G^{(\alpha)})$.

Theorem (M., Patey)

If B is not Δ^1_1, for every covering $A^0 \cup A^1 \supseteq \omega$. If G is sufficiently generic then B is not $\Delta^1_1(G)$ (with in particular $\omega_1^G = \omega_1^{ck}$).

Question

Theorem (Wang)

Let X be non-computable. Let $c : \mathbb{N}^2 \to \{0,1,2\}$ be any coloring. Then there exists G and $i \in \{0,1,2\}$ such that

- *For all $n, m \in G$ we have $c(n,m) \neq i$.*
- *G does not compute X*

Theorem (Cholak, Patey)

Let X be non-computable. Let n and $m > d_n$. Let $c : \mathbb{N}^n \to \{0, m-1\}$ be any coloring. Then there exists G such that

- *$\#\{i \ : \ c(a,b) = i \text{ for } a, b \in G\} \leqslant d_n$.*
- *G does not compute X*

where $d_0 = 1$ and $d_n = \sum_{i=0}^{n} d_i d_{n-1}$ are the Catalan numbers

Question

Can we iterate this (with maybe different numbers) through the jumps ?