Ramsey-like theorems and moduli of computation

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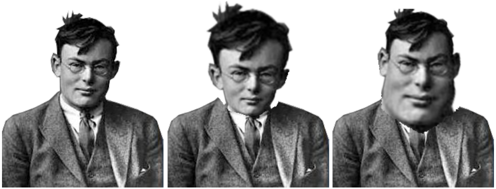
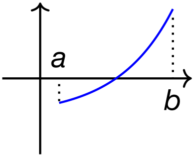
Joint work with Peter Cholak



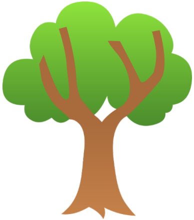
November 27, 2019

Consider mathematical problems

Intermediate value theorem
For every continuous function f over an interval $[a, b]$ such that $f(a) \cdot f(b) < 0$, there is a real $x \in [a, b]$ such that $f(x) = 0$.



König's lemma
Every infinite, finitely branching tree admits an infinite path.



What sets can problems encode?

Fix a problem P.

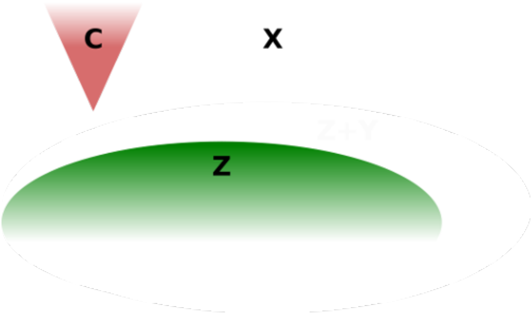
A set S is P-encodable if there is an instance of P such that every solution computes S.

Every computable set is P-encodable.

What sets can problems encode?

Defi (Strong avoidance of 1 cone)

For every Z, every $C \not\leq_T Z$ and every instance X, there is a solution Y such that $C \not\leq_T Z \oplus Y$.



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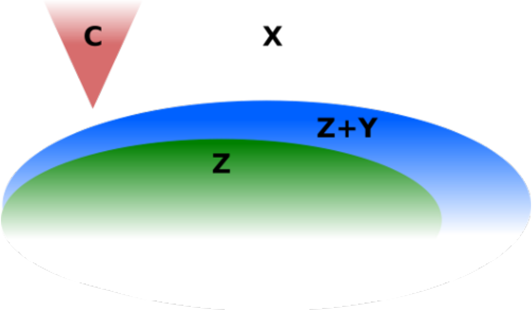


What functions can problems dominate?

Fix a problem P.

A function $f : \omega \to \omega$ is P-dominated if there is an instance of P such that every solution computes a function dominating f

What functions can problems dominate?

A function f is hyperimmune if it is not dominated by any computable function.

Defi (Strong preservation of 1 hyperimmunity)

For every Z, every Z-hyperimmune function f and every instance X, there is a solution Y such that f is $Z \oplus Y$-hyperimmune.

Thm (Downey, Greenberg, Harrison-Trainor, P, Turetsky)

Strong avoidance of 1 cone if and strong preservation of 1 hyperimmunity are equivalent.

Not equivalent in the unrelativized version!

- Fix a non-zero set Y of hyperimmune-free degree.
  Let $\mathsf{P}_1 : Y \mapsto \{Y\}$.
- Fix a hyperimmune f below a Δ^1_1-random.
  Let $\mathsf{P}_2 : f \mapsto \{g : g \geq f\}$.

What sets can encode Ramsey's theorem?

Ramsey's theorem

$[X]^n$ is the set of unordered n-tuples of elements of X

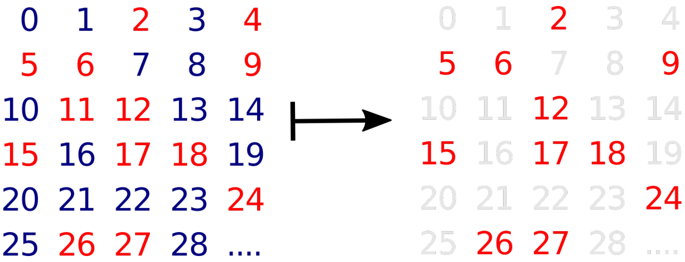
A k-coloring of $[X]^n$ is a map $f : [X]^n \to k$

A set $H \subseteq X$ is homogeneous for f if $|f([H]^n)| = 1$.

RT^n_k Every k-coloring of $[\mathbb{N}]^n$ admits an infinite homogeneous set.

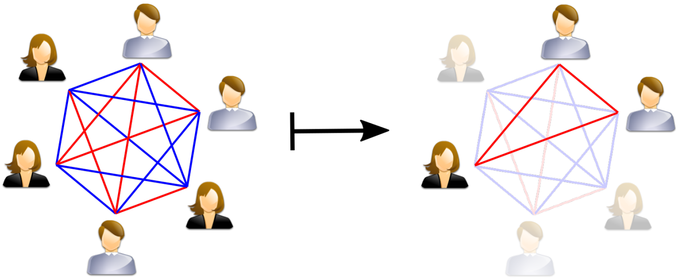
Pigeonhole principle

RT^1_k Every k-partition of $\mathbb{N}$ admits an infinite part.



Ramsey's theorem for pairs

RT^2_k Every k-coloring of the infinite clique admits an infinite monochromatic subclique.



Thm (Jockusch)

Every function is RT^2_2-dominated.

Given $g : \omega \to \omega$, an interval $[x, y]$ is g-large if $y \geq g(x)$.
Otherwise it is g-small.

$$f(x, y) = \begin{cases} 1 & \text{if } [x, y] \text{ is g-large} \\ 0 & \text{otherwise} \end{cases}$$

A function f is a modulus of a set S if every function dominating f computes S.

Thm (Groszek and Slaman)

The sets admitting a modulus are the Δ^1_1 sets.

Thm (Jockusch)

Every Δ^1_1 set is RT^2_2-encodable.

A set S is computably encodable if for every infinite set X, there is an infinite subset $Y \subseteq X$ computing S.

Thm (Solovay)

The computably encodable sets are the Δ^1_1 sets.

Thm (Jockusch)

A set is RT^n_k-encodable for some $n \geq 2$ iff it is Δ^1_1.

The encodability power
of RT^n_k comes from the

sparsity

of its homogeneous sets.

Thm (Dzhafarov and Jockusch)

The RT^1_2-encodable sets are the computable sets.

0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
20	21	22	23	24
25	26	27	28	....

Sparsity of red implies non-sparsity of blue and conversely.

Ramsey's theorem

Over n-tuples

$$\mathsf{RT}^{n}_{k}$$

Using k colors

Ramsey's theorem

Over n-tuples

$$\mathrm{RT}^n_{k,r}$$

Allows r colors

Using k colors

Thm (Wang)

A set is $\mathrm{RT}^n_{k,\ell}$-encodable iff it is computable for large ℓ

(whenever ℓ is at least the nth Schröder Number)

Thm (Dorais, Dzhafarov, Hirst, Mileti, Shafer)

A set is $\mathrm{RT}^n_{k,\ell}$-encodable iff it is Δ^1_1 for small ℓ

(whenever $\ell < 2^{n-1}$)

Thm (Cholak, P.)

Every function is $\mathsf{RT}^n_{k,\ell}$-dominated for $\ell < 2^{n-1}$.

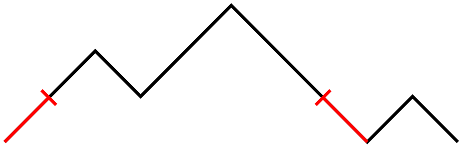
$$f(x_1, x_2, \ldots, x_n) = \langle [x_1, x_2] \text{ g-large?}, \ldots, [x_{n-1}, x_n] \text{ g-large?} \rangle$$

Thm (Cholak, P.)

If a set is $\mathsf{RT}^n_{k,\ell}$-encodable for $\ell \geq 2^{n-1}$ then it is arithmetical.

Catalan numbers

C_n is the number of trails of length $2n$.



$$C_0 = 1 \qquad \text{and} \qquad C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$$

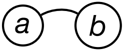
1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786,...

Left-c.e. function

Defi

A function $g : \omega \to \omega$ is left-c.e. if there is a uniformly computable sequence of functions $g_0 \leq g_1 \leq \dots$ limiting to g.

Given $x_0, \dots, x_{n-1}$, define the graph of size n by



- if $b = a + 1$ and $[x_a, x_{a+1}]$ is g-large ; or
- if $b > a + 1$ and $[x_a, x_{a+1}]$ is g_{x_b}-small

Defi

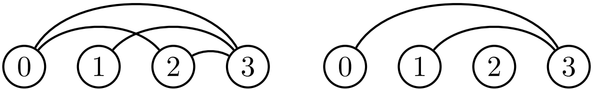
A largeness graph is a pair $(\{0, \dots, n-1\}, E)$ such that

(a) If $\{i, i+1\} \in E$, then for every $j > i+1$, $\{i, j\} \notin E$

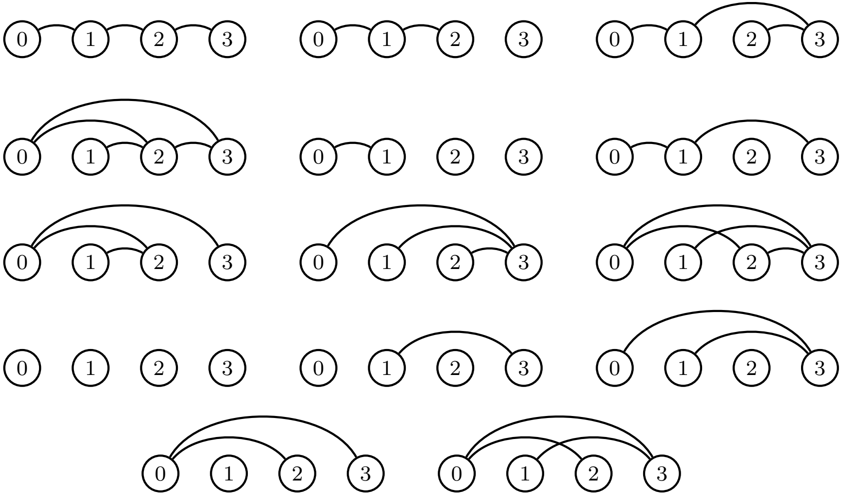
(b) If $i < j < n$, $\{i, i+1\} \notin E$ and $\{j, j+1\} \in E$, then $\{i, j+1\} \in E$

(c) If $i+1 < j < n-1$ and $\{i, j\} \in E$, then $\{i, j+1\} \in E$

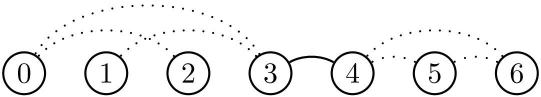
(d) If $i+1 < j < k < n$ and $\{i, j\} \notin E$ but $\{i, k\} \in E$, then $\{j-1, k\} \in E$



Largeness graphs of size 4



Counting largeness graphs



A largeness graph $\mathcal{G} = (\{0, \dots, n-1\}, E)$ is packed if for every $i < n-2$, $\{i, i+1\} \notin E$.

- L_n = number of largeness graphs of size n
- P_n = number of packed largeness graphs of size n

$$L_0 = 1 \qquad \text{and} \qquad L_{n+1} = \sum_{i=0}^{n} P_{i+1} L_{n-i}$$

Counting packed largeness graphs

A largeness graph $\mathcal{G} = (\{0, \dots, n-1\}, E)$ of size $n \geq 2$ is normal if $\{n-2, n-1\} \in E$.

Thm (Cholak, P.)

The following are in one-to-one correspondance:

(a) packed largeness graphs of size n

(b) normal largeness graphs of size n

(c) largeness graphs of size $n-1$

Thm (Cholak, P.)

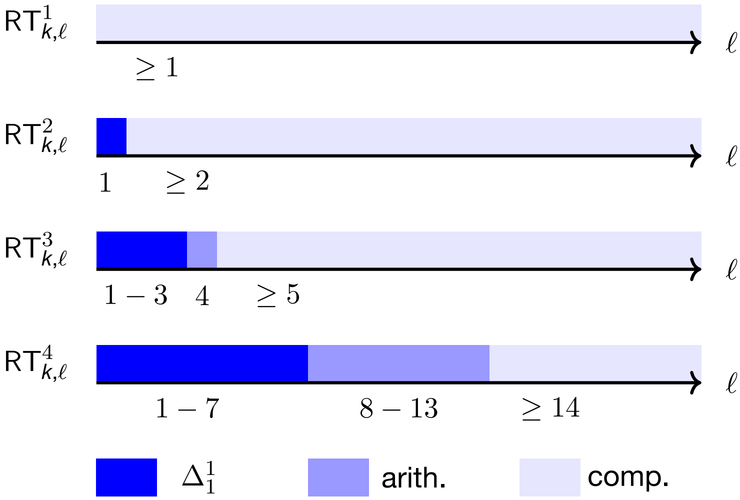
Every left-c.e. function is $\mathsf{RT}^n_{k,\ell}$-dominated for $\ell < C_n$.

$$f(x_1, x_2, \ldots, x_n) = \text{ the largeness graph of g}$$

Thm (Cholak, P.)

The $\mathsf{RT}^n_{k,\ell}$-encodable sets for $\ell \geq C_n$ are the computable sets.

$\mathsf{RT}^n_{k,\ell}$-encodable sets



Ramsey-like theorems

Erdős-Moser theorem

Fix $f : [\omega]^2 \to 2$.

A set H is transitive if for every $a < b < c \in H$, such that $f(a,b) = f(b,c)$ then $f(a,b) = f(a,c)$.

EM Every 2-coloring of $[\mathbb{N}]^2$ admits an infinite transitive set.

Thm (Jockusch)

Every function is RT^2_2-dominated.

Thm (P.)

EM admits strong avoidance of 1 cone.

Is there a maximal weakening of RT^n_k which admits strong avoidance of 1 cone?

Ramsey-like problems

Fix a formal coloring $\mathtt{f} : [\omega]^n \to k$ and variables $\mathtt{x}_0 < \mathtt{x}_1 < \dots$

An RT^n_k-pattern P is a finite conjunction of formulas

$$\mathtt{f}(\mathtt{x}_{i_1}, \dots \mathtt{x}_{i_n}) = v_1 \wedge \dots \wedge \mathtt{f}(\mathtt{x}_{j_1}, \dots \mathtt{x}_{j_n}) = v_s$$

with $v_1, \dots, v_s < k$

Given a coloring $f : [\omega]^n \to k$, a set $H \subseteq \omega$ f-avoids an RT^n_k-pattern P if $(F, f) \not\models P$ for every finite set $F \subseteq H$.

Ramsey-like problems

Defi

Given a set V of RT^n_k-patterns, $\mathsf{RT}^n_k(V)$ is the problem whose instances are colorings $f : [\omega]^n \to k$ and solutions are sets f-avoiding every pattern in V.

In particular, RT^n_k, $\mathsf{RT}^n_{k,\ell}$ and EM are Ramsey-like problems.

Thm (P.)

For every $n, k \geq 1$, there is a strongest Ramsey-like problem $\mathsf{RT}^n_k(V)$ which admits strong avoidance of 1 cone.

Ramsey-like problems

Given problems P and Q, let $\mathsf{P} \leq_{id} \mathsf{Q}$ if $\operatorname{dom} \mathsf{P} \subseteq \operatorname{dom} \mathsf{Q}$, and for every $X \in \operatorname{dom}(\mathsf{P})$, $\mathsf{Q}(X) \subseteq \mathsf{P}(X)$.

Thm (P.)

There is a Ramsey-like problem $\mathsf{SCA\text{-}RT}^n_k$ such that for every set V of RT^n_k-patterns, $\mathsf{RT}^n_k(V)$ admits strong avoidance of 1 cone iff $\mathsf{RT}^n_k(V) \leq_{id} \mathsf{SCA\text{-}RT}^n_k$.

To decide strong avoidance for $\mathsf{RT}^n_k(V)$, simply check that

$$\bigvee V \to \bigvee V_{\mathsf{SCA\text{-}RT}^n_k}$$

is a tautology.

Example: $\mathsf{SCA\text{-}RT}^2_k$

Defi ($\mathsf{SCA\text{-}RT}^2_k$)

For every coloring $f : [\omega]^2 \to k$, there are two colors $s, \ell < k$ and an infinite set $H \subseteq \omega$ such that

- $f[H]^2 \subseteq \{s, \ell\}$
- $f(x,y) = f(y,z) = s$ iff $f(x,z) = s$ for every $x < y < z \in H$

It looks like over H, there is some function $g : \omega \to \omega$ such that

$$f(x,y) = \begin{cases} \ell & \text{if } [x,y] \text{ is g-large} \\ s & \text{otherwise} \end{cases}$$

This analysis generalizes the following theorems:

- RT^2_2 admits avoidance of 1 cone (Seetapun)
- RT^1_2 admits strong avoidance of 1 cone (Dzhafarov and Jockusch)
- EM admits strong avoidance of 1 cone (P.)
- RT^n_{k,C_n} admits strong avoidance of 1 cone (Cholak and P.)
- FS^n admits strong avoidance of 1 cone (Wang)
- ADS does not admit strong avoidance of 1 cone

Conclusion

Ramsey-type problems compute through sparsity.

The computational properties of Ramsey-type problems are consequences of their combinatorics.

The analysis of Ramsey-like theorems is induced by the exact bound analysis of the thin set theeorems.

References

Peter A. Cholak and Ludovic Patey.
Thin set theorems and cone avoidance.
To appear, 2019.

Ludovic Patey.
Ramsey-like theorems and moduli of computation.
arXiv preprint arXiv:1901.04388, 2019.
To appear.

Matthew Harrison-Trainor Ludovic Patey Rod Downey, Noam Greenberg and Dan Turetsky.
Relationships between computability-theoretic properties of problems.
To appear, 2019.