On the Decidability of Reachability in Linear Time-Invariant Systems

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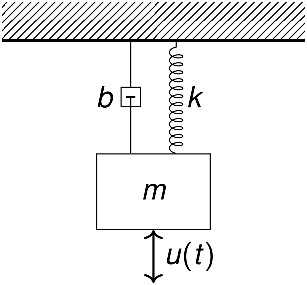
26 november 2019

Example : mass-spring-damper system



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State : $X = z \in \mathbb{R}$

Equation of motion :

$$mz'' = -kz - bz' + mg + u$$

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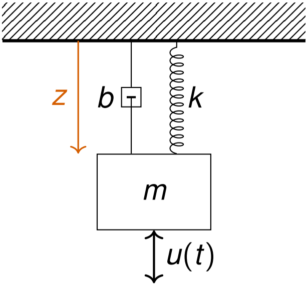

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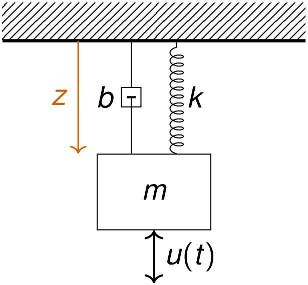

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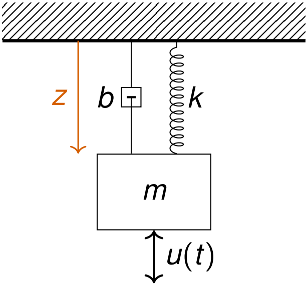

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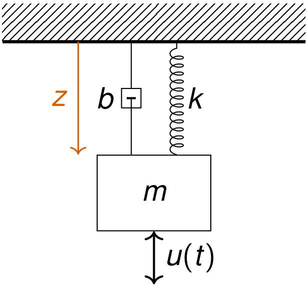

Model with external input $u(t)$
$\rightarrow$ Linear time invariant system

$$X' = AX + Bu$$

with some constraints on u.

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Linear dynamical systems

Discrete case

$$x(n+1) = Ax(n)$$

- biology,
- software verification,
- probabilistic model checking,
- combinatorics,
- ....

Continuous case

$$x'(t) = Ax(t)$$

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- physics,
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Typical questions

- reachability
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- optimal control
- feedback control
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The problem

LTI-REACHABILITY

- a source $s \in \mathbb{Q}^d$,
- a target $t \in \mathbb{Q}^d$,
- a transition matrix $A \in \mathbb{Q}^{d \times d}$,
- a set of controls $U \subseteq \mathbb{R}^d$,

decide if $\exists T \in \mathbb{N}, u_0, \ldots, u_{T-1} \in U$ such that $x_T = t$ where

$$x_0 = s, \qquad x_{n+1} = Ax_n + u_n.$$

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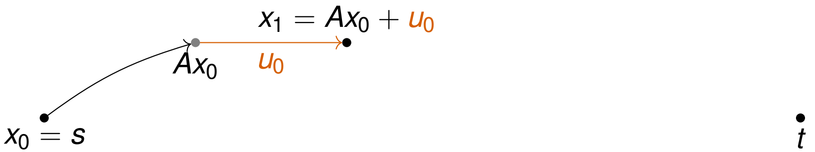

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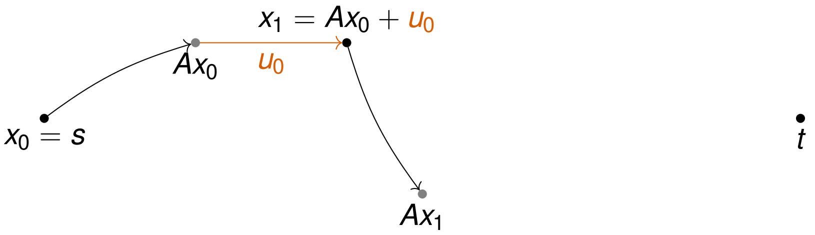

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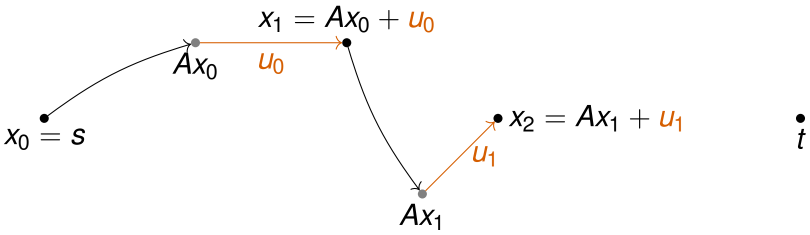

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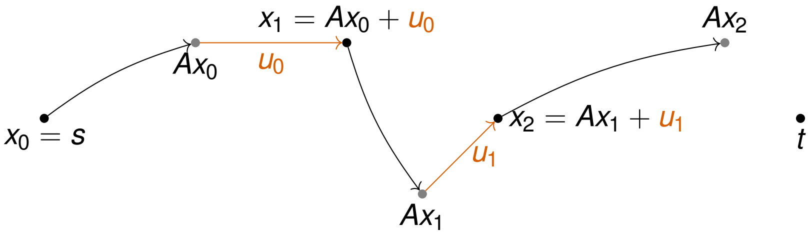

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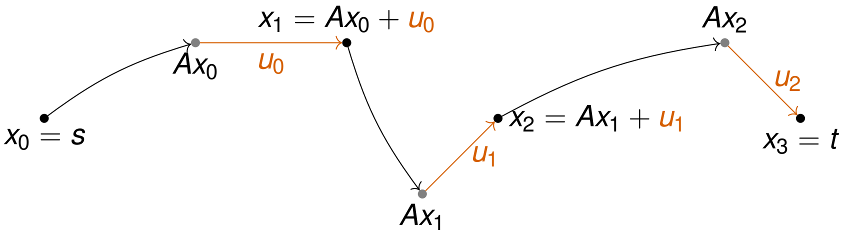

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Almost no exact results for other classes of U in particular when U is bounded (which is the most natural case).

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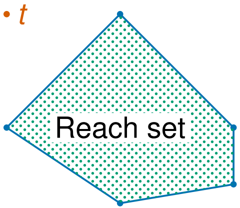

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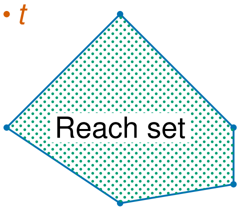

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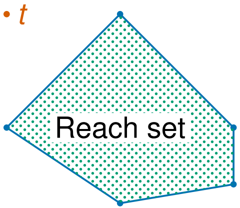

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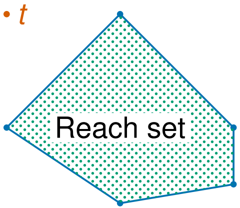

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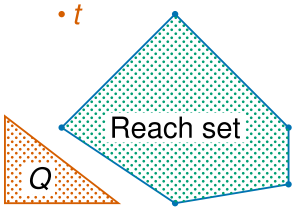
Remark : in fact we can decide reachability to a convex polytope Q.



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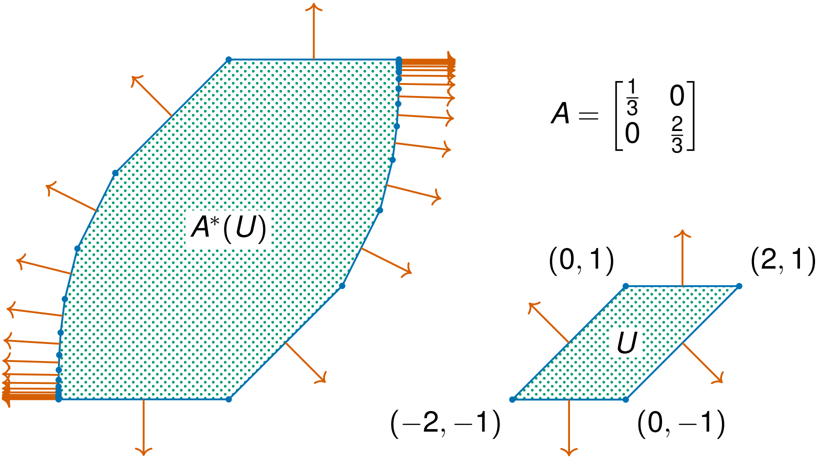
Why is this problem hard

The reachable set $A^*(U)$ can have **infinitely** many faces.



$$A = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{bmatrix}$$

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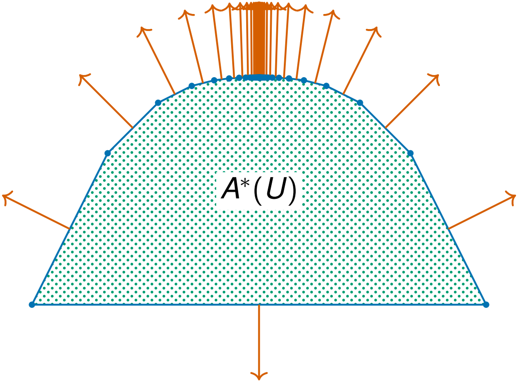
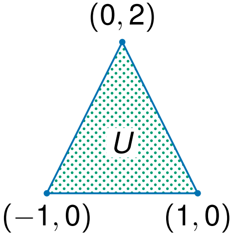
The reachable set $A^*(U)$ can have **faces of lower dimension** : the "top" extreme point does not belong to any facet.



$$A = \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$



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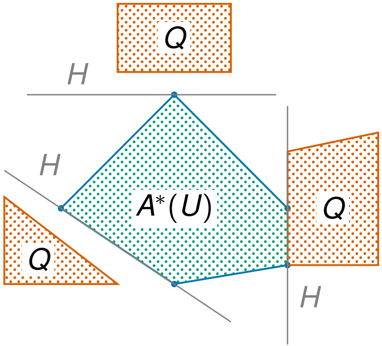

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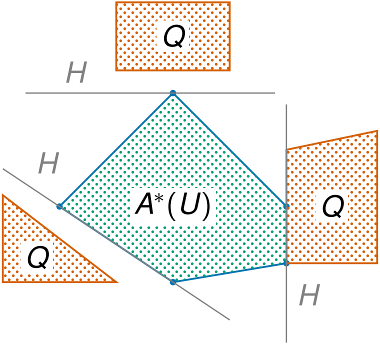

Further difficulty : a separating hyperplane may not be supported by a facet of either $A^*(U)$ or Q.

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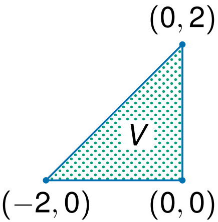
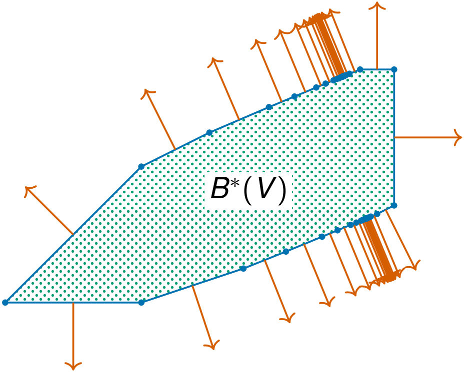
$$B = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 \end{bmatrix}$$





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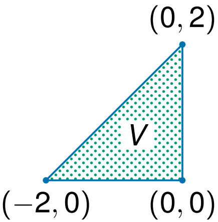

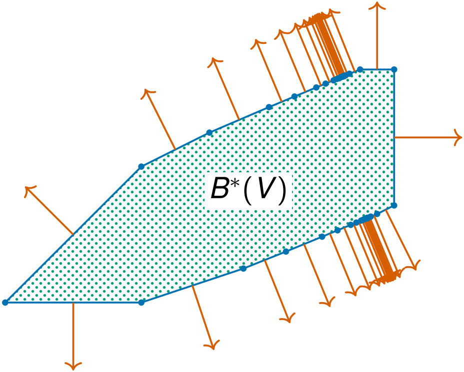

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Theorem (Non-reachable instances)

There is a separating hyperplane with algebraic coefficients.

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$$A = \frac{1}{2}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad U = [0,1] \times \{0\}.$$

Decidability of $t \leqslant \sum_{n=0}^{\infty} \max(0, 2^{-n}\cos(n\theta))$ unknown.

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Exact reachability of $x_{n+1} = Ax_n + u_n$:

- decidability crucially depends on the shape of the control set
- even with convex bounded inputs, the problem is very hard (Skolem/Positivity, open for 70 years)
- we can recover decidability using strong spectral assumptions

Open questions :

- for convex bounded inputs, is it Positivity-easy ?
- weaken spectral assumptions ? Minimal difficult example :

$$A = \frac{1}{2}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad U = [0,1] \times \{0\}.$$

Decidability of $t \leqslant \sum_{n=0}^{\infty} \max(0, 2^{-n}\cos(n\theta))$ unknown.

Work in progress : continuous case $x'(t) = Ax(t) + u(t)$ Details

Backup slides

Continuous control

Rinse and repeat :

$$x'(t) = Ax(t) + u(t)$$

where $u : \mathbb{R} \to U$ measurable.

Problems : reachability, safety, controllability, ...

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It looks similar but

- basic questions look harder :

$$x(t) = e^{At}x_0 + \int_0^t e^{-As}u(s)\,\mathrm{d}s.$$

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- harder questions look easier :

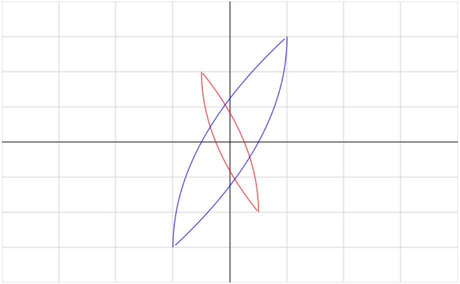
linear $+$ continuous $=$ hard to encode problems

Continuous control : preliminary results

Theorem (Joint work with Mohan Dantam, preliminary)

Point-to-point continuous control is

- *decidable in dimension* 2*,*
- *conditionally decidable with real eigen values,*
- *conditionally decidable in bounded time,*
- *Skolem/Positivity hard for point-to-set.*



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