random bits in practice and theory

RaCAF project

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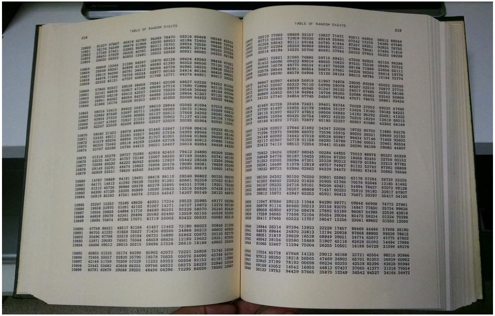
Is randomness real?

randomness around us

more serious efforts

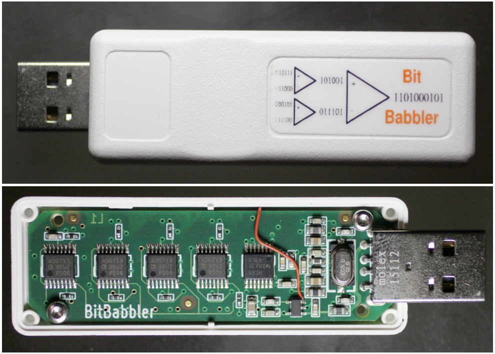


2 TABLE OF RANDOM DIGITS

00050	09188	20097	32825	39527	04220	86304	83389	87374	64278	58044
00051	90045	85497	51981	50654	94938	81997	91870	76150	68476	64659
00052	73189	50207	47677	26269	62290	64464	27124	67018	41361	82760
00053	75768	76490	20971	87749	90429	12272	95375	05871	93823	43178
00054	54016	44056	66281	31003	00682	27398	20714	53295	07706	17813

Rand Corporation, *A Million Random Digits with 100,000 Normal Deviates* (1955)

electronic devices



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- still the choice of programming language in advance is more reasonable than the choice of the test

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- exist iff one-way functions exist (Hastad, Impagliazzo, Luby, Levin)

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- also two independent weakly random sources

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- cryptographic protocols (one-time pad, secret sharing)

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- a special type of "whitening": no hope to get uniform randomness, just computably indistinguishable

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- conjecture: digits of π form a normal sequence

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- secondary tests (in Knuth, widely used in diehard)

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It is quite straightforward to define whether a sequence of infinite length is random or not. This sequence is random if the quantity of information it contains – in the sense of Shannon's information theory – is also infinite.

In other words, it must not be possible for a computer program, whose length is finite, to produce this sequence. Interestingly, an infinite random sequence contains all possible finite sequences.

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- (making the last statement false)

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- "For a *P-value* $\geqslant 0.001$, a sequence would be considered to be random with a confidence of 99.9%. For a *P-value* < 0.001, a sequence would be considered to be non-random with a confidence of 99.9%" (1-4)

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- "Unlike α [the probability of Type I error], β is not a fixed value. $\langle \dots \rangle$ The calculation of Type II error β is more difficult than the calculation of α because of the many possible types of non-randomness"
- "If a *P-value* for a test is determined to be equal to 1, then the sequence appears to have perfect randomness" (1-4)
- "For a *P-value* $\geqslant 0.001$, a sequence would be considered to be random with a confidence of 99.9%. For a *P-value* < 0.001, a sequence would be considered to be non-random with a confidence of 99.9%" (1-4)
- two incorrect tests deleted from the second version

theory vs. practice: `diehard[er]`

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- diehard: secondary tests based on incorrect assumptions
- dieharder: "At this point I think there is rock solid evidence that this test [one of the `diehard` tests] is completely useless in every sense of the word. It is broken, and it is so broken that there is no point in trying to fix it. The problem is that the transformation above is not linear, and doesn't work. Don't use it."

theory vs. practice: entropy

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- entropy of a distribution (Shannon)

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- for individual objects: Kolmogorov complexity

theory vs. practice: entropy

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- for individual objects: Kolmogorov complexity
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  "The central mathematical concept underlying this [NIST] Recommendation is entropy. Entropy is defined relative to one's knowledge of an experiment's output prior to observation, and reflects the uncertainty associated with predicting its value – the larger the amount of entropy, the greater the uncertainty in predicting the value of an observation"

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- "Each bit of a bitstring with full entropy has a uniform distribution and is independent of every other bit of that bitstring. Simplistically, this means that a bitstring has full entropy if every bit of the bitstring has one bit of entropy; the amount of entropy in the bitstring is equal to its length' (same NIST document)

theory vs. practice: whitening

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- similar results for k bits: for $F\colon \mathbb{B}^n \longrightarrow \mathbb{B}^k$ there is SV source and some k-bit output string that appear with probability at least $(2/3)^k$ instead of $(1/2)^k$

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- but for fixed R (generated, sent with the device)
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- strong extractor: $(F(X, R), R) \approx$ uniform
- can be saved, but only with half of the security parameter

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- randomness extractors with several independent sources

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- independence is physically plausible

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- tests are hard to debug
- NIST says:
  In practice, many reasons can be given to explain why a data set has failed a statistical test. The following is a list of possible explanations. The list was compiled based upon NIST statistical testing efforts.
  1. An incorrectly programmed statistical test.
  2. An underdeveloped (immature) statistical test.
  3. An improper implementation of a random number generator.
  4. Improperly written codes to harness test input data.
  5. Poor mathematical routines for computing *P-values*.
  6. Incorrect choices for input parameters.

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- may reject a good generator using a bad reference
- $S(x_1),\ldots,S(x_n)$ vs $S(x_{n+1}\oplus y_1),\ldots,S(x_{n+m}\oplus y_m)$

survey of available generators

parameters to take into account:

survey of available generators

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- noise source

survey of available generators

parameters to take into account:

- noise source
- whitening

survey of available generators

parameters to take into account:

- noise source
- whitening
- access to raw noise

survey of available generators

parameters to take into account:

- noise source
- whitening
- access to raw noise
- rate

survey of available generators

parameters to take into account:

- noise source
- whitening
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- cost

survey of available generators

parameters to take into account:

- noise source
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- access to raw noise
- rate
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- software integration

survey of available generators

parameters to take into account:

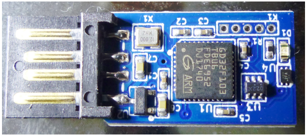
- noise source
- whitening
- access to raw noise
- rate
- cost
- software integration
- bonus: open source hard/software

Araneus

$$$, zener noise, 100 kbits/s, raw=no, whitening=?

"The raw output bits from the A/D converter are then further processed by an embedded microprocessor to combine the entropy from multiple samples into each final output bit, resulting in a random bit stream that is practically free from bias and correlation"

Gniibe

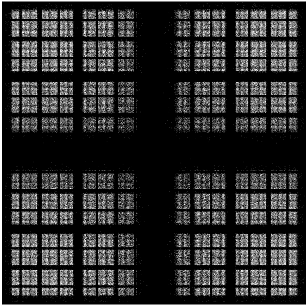
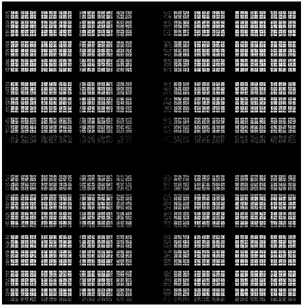


$$, environment noise, 3 mbits/s, access to raw bits, open source (based on GNU microprocesssor unit), whitening=CRC32 + SHA-256

Infinite Noise

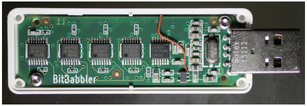
\$\$, electronic noise, $x \longmapsto 2x - 1$ digitization, 300 kbits/s, access to raw bits, whitening=SHA3

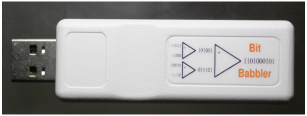
analysis of raw noise bits

infinite noise: measured vs. model

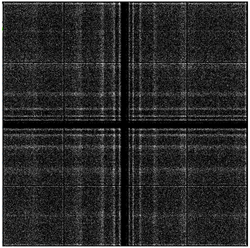
Bitbabbler



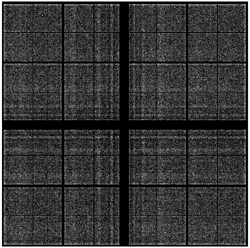


\$\$–\$\$\$, electronic noise, $x \longmapsto 2x - 1$ digitization, 2.5 mbits/s default, 4 independent generators (\$150 version), access to raw bits, variable discretization rate, whitening=XOR

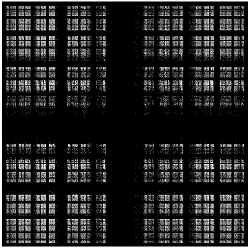
Bitbabbler: changing rate



100 kHz

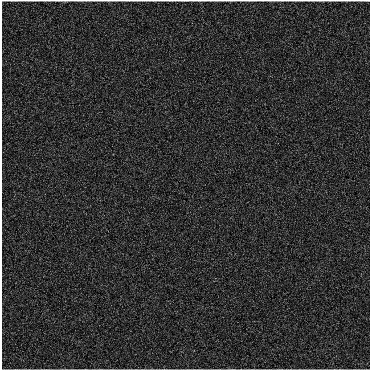


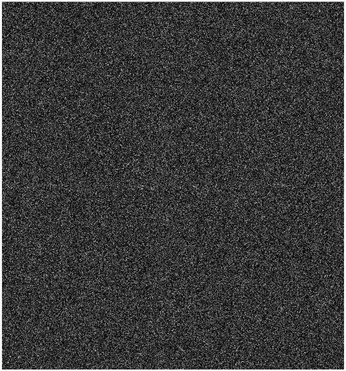
default rate 2.5 MHz



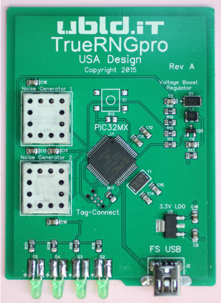
5 MHz

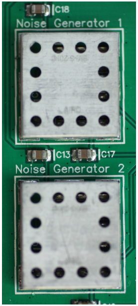
2 or3 XOR





TrueRNG



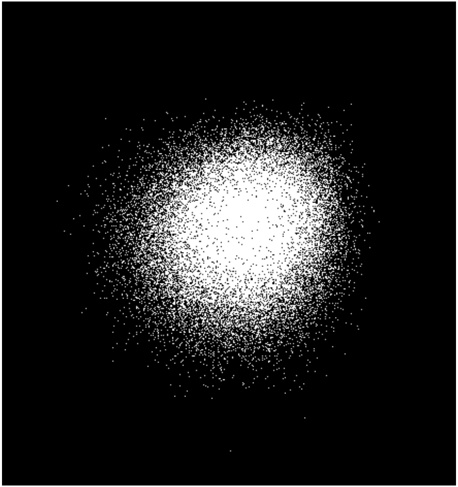


$$–$$$, zener noise + ADC,

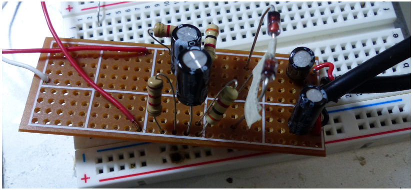
3.2 mbits/s, 2 independent generators ($100 version),

access to raw bits, whitening=XOR/CRC

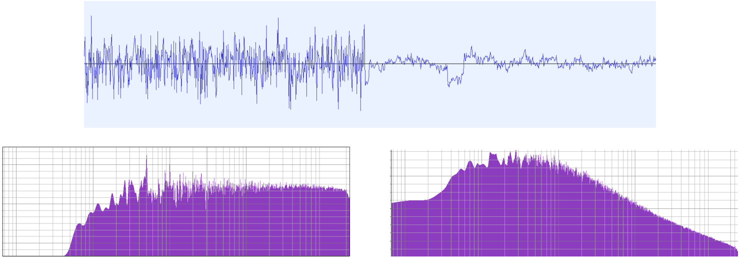
TrueRNG raw noise



DIY approach



DIY: not all noise sources are the same

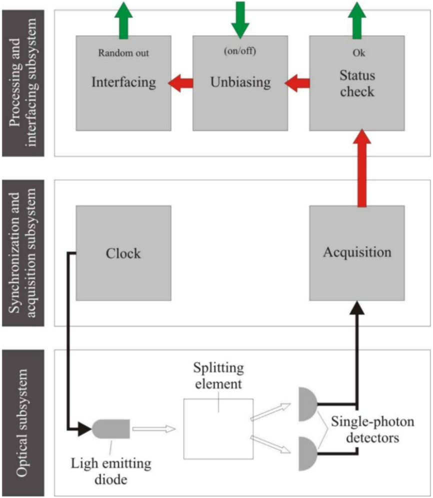


two zener diodes from the same roll

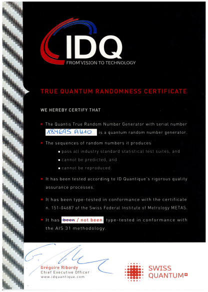
ID Quantique

$$$–$$$$, photon detectors, 4 mbits/s, no access to raw bits, whitening?, additional randomness extraction available

ID Quantique: scheme



certificates as randomness theater?



IDQ
FROM VISION TO TECHNOLOGY

TRUE QUANTUM RANDOMNESS CERTIFICATE

WE HEREBY CERTIFY THAT

- The Quantis True Random Number Generator with serial number 184695 A110 is a quantum random number generator.
- The sequences of random numbers it produces
  - pass all industry standard statistical test suites, and
  - cannot be predicted, and
  - cannot be reproduced.
- It has been tested according to ID Quantique's rigorous quality assurance processes.
- It has been type-tested in conformance with the certificate n. 151-04687 of the Swiss Federal Institute of Metrology METAS.
- It has ~~been~~ / not been type-tested in conformance with the AIS 31 methodology.

Grégoire Ribordy
Chief Executive Officer
www.idquantique.com

SWISS
QUANTUM

still fails dieharder/ent tests (before optional randomness extractor)

security through obscurity

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- “approved hash function”

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- and even it were, it won’t help

NIST says:

Hash_DRBG's [the random generator based on hash functions] security depends on the underlying hash function's behavior when processing a series of sequential input blocks. If the hash function is replaced by a random oracle, Hash_DRBG is secure. It is difficult to relate the properties of the hash function required by Hash_DRBG with common properties, such as collision resistance, pre-image resistance, or pseudorandomness.

vulnerabilities

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- software attack if a microprocessor is used

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- last but not least: stupid errors (e.g., AMD Zen FF random generator)

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THANKS!