Admissibles and computations

Bruno Durand Grégory Lafitte

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Quelqu'un va-t-il prendre enfin la défense de l'infini ?
—Aragon, Paris-Journal, 13 avril 1923

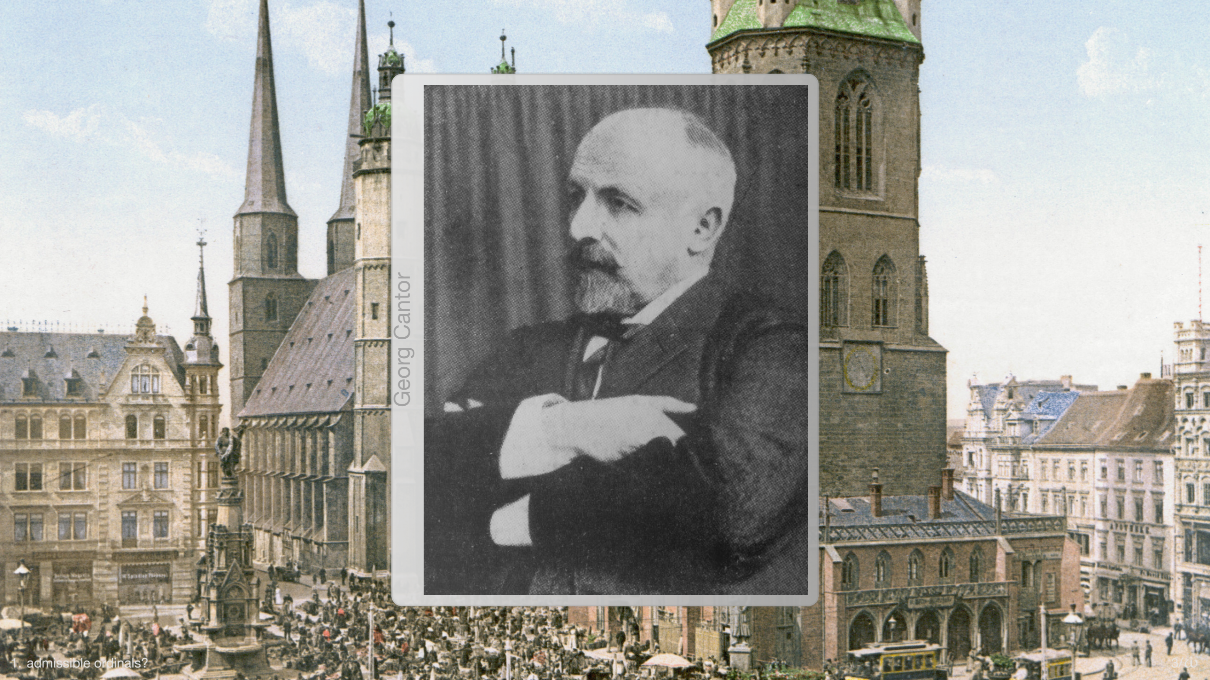
AGENCE NATIONALE DE LA RECHERCHE ANR

1. admissible ordinals?

Cantor a trouvé une loi d'engendrement de la multitude des nombres ordinaux finis et transfinis, il a trouvé une dynastie, celle des Aleph, et cela, à l'aide de deux principes seulement, l'un immanent (additif), l'autre transcendant (passage à la limite) : Cantor, législateur de l'infini.

—Sinisgalli, Horror vacui

1. admissible ordinals?



Georg Cantor

Ordinals, a whole bunch of them

DEF Equivalence class of well-orderings by isomorphism

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DEF Transitive set well-ordered by $\in$ 

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Ordinals + transfinite induction
measuring of provability strength

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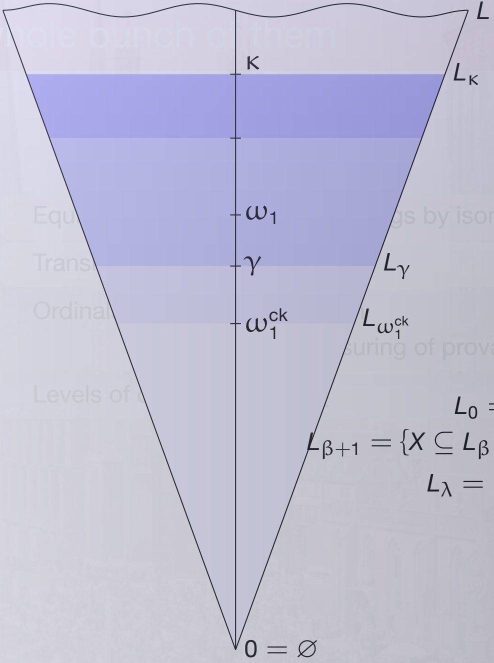
Levels of constructibility L_α

Constructible Universe



$$L_0 = \varnothing$$

$$L_{\beta+1} = \{X \subseteq L_\beta \text{ def.}\} = \mathrm{Def}(L_\beta)$$

$$L_\lambda = \bigcup_{\beta<\lambda} L_\beta$$

Ordinals, a whole bunch of them

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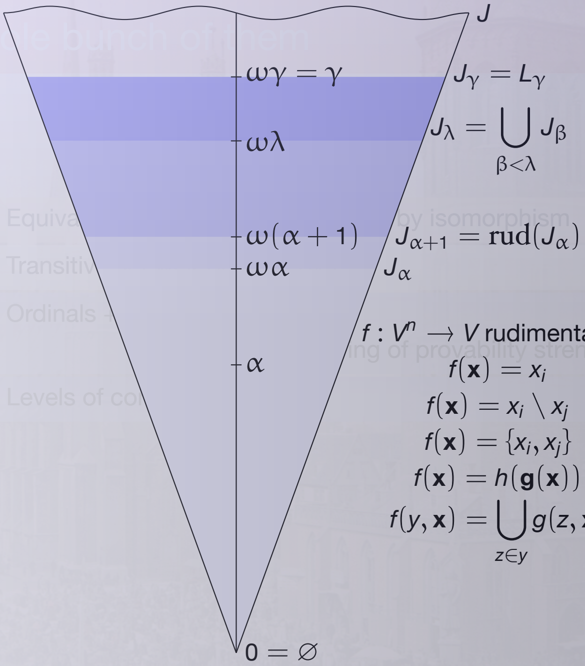

$f : V^n \longrightarrow V$ rudimentary if

$$f(\mathbf{x}) = x_i$$
$$f(\mathbf{x}) = x_i \setminus x_j$$
$$f(\mathbf{x}) = \{x_i, x_j\}$$
$$f(\mathbf{x}) = h(\mathbf{g}(\mathbf{x}))$$
$$f(y, \mathbf{x}) = \bigcup_{z \in y} g(z, \mathbf{x})$$

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Coding infinite ordinals

Ordinals, a whole bunch of them

$\langle x, y \rangle = 1$ iff $x < y$ ($\langle \cdot, \cdot \rangle$ codage de $\mathbb{N}^2$ dans $\mathbb{N}$)

ordre sur ω

coding

ordre sur ω *vs* ordre sur une partie de ω
(soit n est comparable à une infinité d'entiers, soit n est isolé)

si $\forall x \; \langle n, x \rangle = 0$, alors soit n est le plus petit élément, soit il n'est pas dans l'ordre.
si de plus $\forall x \; \langle x, n \rangle = 0$ alors x n'est pas dans l'ordre

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DEF An ordinal is recursive if it has a recursive coding on ω

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If α has a recursive coding on $E \subset \omega$ then it is recursive

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A code for $\beta < \alpha$ extracted from a code for α


Indiana University


Jon Barwise

Admissible ordinals

closed enough ordinals | limit, limit of limits, *etc*

Admissible ordinals

closed enough ordinals — limit, limit of limits, *etc*

DEF ω_1^{CK} is the sup of the recursive ordinals

Admissible ordinals

Σ_0-separation:
for any set E, Σ_0 formula ϕ,
there exists $X \subseteq E$ such that
$$x \in X \iff \phi(x)$$

Σ_0-collection:
for any Σ_0 formula $\phi(x,y)$ s.t. $\forall x \exists y\, \phi(x,y)$, we have that $\forall X \exists E$ s.t. $[e \in E \iff \exists x \in X\, \phi(x,e)]$

closed enough ordinals — limit, limit of limits, *etc*

DEF ω_1^{CK} is the sup of the recursive ordinals

DEF α is *admissible* if L_α is a model of KP 

KP:
- extensionality
- induction
- empty set
- pairing
- union
- Σ_0-separation
- Σ_0-collection

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α is admissible if limit and $L_\alpha \models \Sigma_0$-collection

KP $\begin{cases} \text{extensionality} \\ \text{induction} \\ \text{empty set} \\ \text{pairing} \\ \text{union} \\ \Sigma_0\text{-separation} \\ \Sigma_0\text{-collection} \end{cases}$

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THM $\omega_1^{CK,r}$ is admissible for every real r

KP $\begin{cases} \text{extensionality} \\ \text{induction} \\ \text{empty set} \\ \text{pairing} \\ \text{union} \\ \Sigma_0\text{-separation} \\ \Sigma_0\text{-collection} \end{cases}$

2. gaps?

ITTMs, clockable ordinals, gaps.

ITTM: clockable ordinals and gaps

DEF

Infinite time Turing machines (ITTM)

$\limsup$ special limit *state* *head* limit behaviour

can compute on *reals*

ITTM: clockable ordinals and gaps

ITTMs and what not

ITTMs are extensions of Turing machines to transfinite time.

For simplicity: alphabet $\{0, 1\}$

Ordinal stages

- *successor* ordinals: it works exactly as a Turing machine
- *limit* ordinals: each cell is set to the $\limsup$ of its values, head is rewinded back to the origin and the machine enters a special limit state *L*

3 tapes: input, scratch, output

A real (infinite binary string) can be considered as input (*oracle* computation) and output

Example: a coding of an ordinal may be written, or taken as input by an ITTM

ITTM: clockable ordinals and gaps

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α countable is *writable* if r_α is

ITTM: clockable ordinals and gaps

Infinite time Turing machines (ITTM)

coding ω_1^{CK}

Any recursive ordinal α can be finitely represented by a Turing Machine μ such that $\mu(i) = r_\alpha(i)$

Among these TM we choose one (e.g. of smallest index) $\alpha \mapsto \mu_\alpha$

Our order (partial) : $\mu_\alpha \prec \mu_\beta \iff \alpha < \beta$

This order $\prec$ is not recursive and is of type ω_1^{CK}

This order does not exhibit a recursive minimality — we will improve it using ITTMs

ITTM: clockable ordinals and gaps

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α countable is *clockable* if $\exists$ ITTM which halts exactly after α many steps

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λ_∞ = sup of writables γ_∞ = sup of clockables

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λ_∞ = sup of writables γ_∞ = sup of clockables

DEF

eventually writable ordinals ζ_∞ = sup

accidentally writable ordinals Σ_∞ = sup

ITTM: clockable ordinals and gaps

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DEF $\lambda_\infty =$ sup of writables $\gamma_\infty =$ sup of clockables

DEF eventually writable ordinals $\zeta_\infty =$ sup

accidentally writable ordinals $\Sigma_\infty =$ sup

THM There are non-clockable writable ordinals

Clockable ordinals and gaps

THM Admissibles are not clockable

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THM ω_1^{CK} starts a gap of length ω

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$\gamma_\infty = \lambda_\infty$

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- How are gaps distributed?
- Is there a link between a gap's size and its starting point?

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- Is there a gap with exactly one admissible ordinal properly inside?

Admissibles are not clockable

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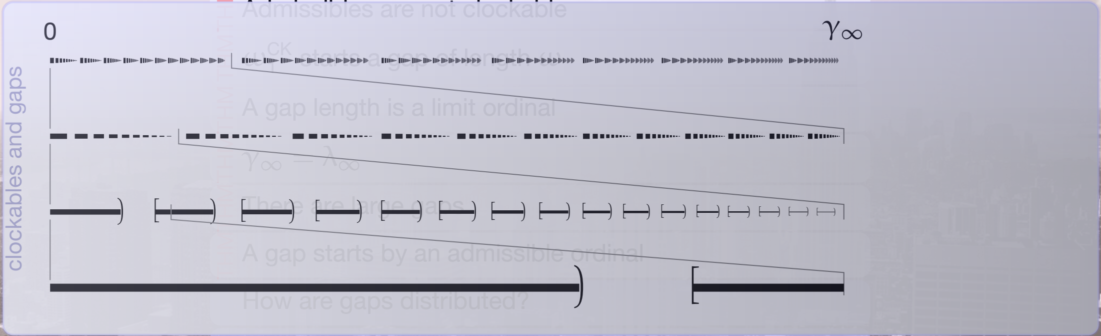
How are gaps distributed?



Is there a link between a gap's size and its starting point?

Is there a gap with exactly one admissible ordinal properly inside?

3. a gap with exactly one admissible

One admissible

THM $L_{\lambda_\infty} \prec_1 L_{\zeta_\infty} \prec_2 L_{\Sigma_\infty}$ 

One admissible

CORTHM

$L_{\lambda_\infty} \prec_1 L_{\zeta_\infty} \prec_2 L_{\Sigma_\infty}$

There is a gap with at least one admissible inside

One admissible

using λ-ζ-Σ

There are many admissible ordinals between the sup of writables λ_∞ and $\zeta_\infty < \Sigma_\infty$.

$$\exists \alpha < \beta < \gamma \text{ s.t. } \beta \text{ adm. and } L_\gamma \text{ witnesses } [\alpha, \beta) \text{ contains no clockable}$$

This is a Σ_1 statement, verified in L_{Σ_∞}, and thus also in L_{λ_∞}. Let $\alpha, \beta, \gamma < \lambda_\infty$ witness this. L_γ believes that β is an admissible ordinal properly inside a gap.

Since any ITTM-computation on the empty input of length $< \beta$ is already contained in L_β, $[\alpha, \beta)$ is indeed inside a gap which properly contains the admissible ordinal β.

One admissible

algorithmic variant

Design an algorithm that checks if a real x is a code for an ordinal which is the starting point of a gap containing $\omega_1^{\mathrm{CK},x}$.

This algorithm accepts any code for λ_∞.

There is an x that is accepted by this algorithm

is a Σ_1 statement.

$L_{\lambda_\infty} \prec_1 L_{\zeta_\infty} \prec_2 L_{\Sigma_\infty}$

There is a gap with at least one admissible inside

By the $\lambda - \zeta - \Sigma$ theorem, we have a witness of this property in L_{λ_∞}, which has to be the code of an admissible ordinal $< \lambda_\infty$ beginning a gap with an admissible properly inside.

One admissible

THM $L_{\lambda_\infty} \prec_1 L_{\zeta_\infty} \prec_2 L_{\Sigma_\infty}$

COR There is a gap with at least one admissible inside

THM The first gap with at least an admissible inside will only have one admissible τ, and ends at $\tau + \omega$

One admissible

Algorithm :

Primary gap detection : we run a universal online-simulation of all ITTM in order to detect gaps. We thus observe gaps ω steps after their starting point.

At each starting points α we have at our disposal a coding of α.

We then start a *secondary gap detection* with oracle α until either :

proof

- the primary gap ends; we then stop secondary detection – we continue the primary detection until next gap
- the first secondary gap is observed; we then halt.

Proof :

If no admissible inside a gap, then it halts after λ_∞. Contradiction.

Halting time: $\alpha + \omega_1^{\mathrm{CK},\alpha} + \omega = \omega_1^{\mathrm{CK},\alpha} + \omega$. Hence this is the end of the gap.

4. beyond one admissible

Several admissibles, ranks, large gaps

THM The first gap with at least $n < \omega$ admissibles inside will only have n admissibles inside, and ends ω steps after the last one

Several admissibles, ranks, large gaps

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The first gap with at least $n < \omega$ admissibles inside will only have n admissibles inside, and ends ω steps after the last one

proof

Direct adaptation of previous proof

Gap : $[\alpha_0, \ldots, \alpha_1, \ldots, \ldots, \ldots, \alpha_n]$

Computation of ω_1^{CK} relativized to detect the next admissible :

$$\alpha_{i+1} = \omega_1^{\mathrm{CK}, \alpha_i}$$

Several admissibles, ranks, large gaps

THM The first gap with at least $n < \omega$ admissibles inside will only have n admissibles inside, and ends ω steps after the last one

THM ($\alpha < \lambda_\infty$) After the writing time for α, the first gap with at least α admissibles inside has exactly α inside

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proof Not a direct adaptation of previous proof because of limits of admissibles

Problem: if $\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots \to \alpha_\omega$ then sometimes α_ω is admissible (*recursively inaccessible*) and sometimes not.

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DEF Ranks

rank-0: admissibles

rank-$\alpha + 1$: admissible limits of rank α adm.

rank-λ: adm. limits of rank β adm. $\forall \beta < \lambda$

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THM Let $\alpha < \lambda_\infty$. After the writing time for α, the first gap starting with a rank-α admissible is of size ω

Several admissibles, ranks, large gaps

THM: The first gap with at least $n < \omega$ admissibles inside will only have n admissibles inside, and ends ω steps after the last one

THM: ($\alpha < \lambda_\infty$) After the writing time for α, the first gap with at least α admissibles inside has exactly α inside

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THM: The smallest $\alpha =$ its rank starts a gap of size ω

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THM Let $f : \omega_1 \to \omega_1$ be ITTM-computable. Then there is an ordinal α starting a gap of size $\geqslant f(\alpha)$

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THM ...and a few other improvements

5. sacks' characterizations of *admissible* ordinals

> *Il y a en nous une sensation finie de « l'infini». Et ce n'est qu'un effet – une conséquence. Ce n'est pas une preuve de quoi que ce soit.* —*Paul Valéry, 1910*


MCMXVI
MASSACHVSETTS INSTITVTE OF TECHNOLOGY

Gerald E. Sacks

Characterizations of *admissibles*

THM For every countable admissible α, $\exists r$ s.t. $\alpha = \omega_1^{CK,r}$

Characterizations of *admissibles*

THM For every countable admissible α, $\exists r$ s.t. $\alpha = \omega_1^{\mathsf{CK},r}$

How can we understand & prove this in an explicit way?

Characterizations of *admissibles*

coding ω_1^{CK} and more

simulation of all ITTM's ω-online

ω machines halt for a given simulation time, we choose the first one in the simulation

$\mu \prec \nu \iff (\mu$ and ν are chosen and μ halts before $\nu)$

if we run the above process up to α we get an order for all clockables before α

first run of ω non clockables $\Longrightarrow \omega_1^{CK}$

the order type of clockables below a gap is exactly the starting point of the gap

coding of λ_∞

same with oracle (or input)

For every countable admissible α, $\exists r$ s.t. $\alpha = \omega_1^{CK,r}$

coding of $\omega_1^{CK,r}$

in some sense these are the *simplest* codings

How can we understand & prove this in an explicit way?

Characterizations of *admissibles*

THM For every countable admissible α, $\exists r$ s.t. $\alpha = \omega_1^{\mathrm{CK},r}$

How can we understand & prove this in an explicit way?

THM The writing time of any $\alpha < \lambda_\infty$ is the sup of all ends of gaps that start before α

6. building up to a proof

Qui est là ? Ah très bien : faites entrer l'infini.

—Aragon, Une vague de rêves, 1924

The first steps

Successor admissible case

The first steps

Successor admissible case

σχέδιος

$\alpha = \beta^+$ (α and β are admissible)

Code β in a real r to obtain $\omega_1^{\mathrm{CK},r} = \alpha$

The first steps

Successor admissible case

LEM

If r_β codes an ordinal β, $\omega_1^{\mathrm{CK},r_\beta} \geqslant \beta^+$

The first steps

Successor admissible case

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If r_β codes an ordinal β, $\omega_1^{\mathrm{CK},r_\beta} \geqslant \beta^+$

We have to make sure that r does not code more…

The first steps

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If r_β codes an ordinal β, $\omega_1^{\mathrm{CK},r_\beta} \geqslant \beta^+$

We have to make sure that r does not code more…

Find the lowest simplest code

The first steps

Successor admissible case

σχέδιο

If coding level of r_β is $< \beta^+$, we have equality and everything works.

Otherwise, we need to do something a little bit more sophisticated.

Find the lowest simplest code

The first steps

Successor admissible case

LEM If r_β codes an ordinal β, $\omega_1^{\mathrm{CK},r_\beta} \geqslant \beta^+$

We have to make sure that r does not code more…

Find the lowest simplest code

Recursively inaccessible case

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If r_β codes an ordinal β, $\omega_1^{\mathrm{CK},r_\beta} \geqslant \beta^+$

We have to make sure that *r* does not code more…

Find the lowest simplest code

Recursively inaccessible case

Limit of the ω first admissibles is not admissible

The first steps

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Find the lowest simplest code

Recursively inaccessible case

Limit of the ω first admissibles is not admissible

One can build an ITTM which halts at exactly that limit

The first steps

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Consider the first *recursively inaccessible* ι_0


La Sorbonne

The *lost* lemma

We need some tool to construct r

The *lost* lemma

We need some tool to construct r

LEM $\{r_i : i \in \omega\}$ s.t. $\forall i,\ r_i <_T r_{i+1}$

$\exists r$ s.t. $\forall i,\ r_i \leqslant_T r$ but $\bigoplus_i r_i \nleqslant_T r \leqslant_T (\bigoplus_i r_i)'$

The *lost* lemma

σχέδιο

r needs to code the r_i's but in such a way that $\forall i$, r_i can be computed from r but not uniformly.

r is a *mapping* from ω^2 to $\{0, 1\}$ that contains r_i in the $\mathfrak{b}(i)$ column.

We ensure that $\bigoplus_i r_i \not\leq_T r$ by adding the needed information to hide the r_i's.

r is constructed as $\bigcup_i o_i$ where the o_i's are compatible oracles that represent the left part of r up to column $\mathfrak{b}(i)$

We assume that o_i has been built and that we know $\mathfrak{b}(i)$, and we give the construction for o_{i+1} and $\mathfrak{b}(i+1)$.

The *lost* lemma

σχέδιο

Consider all $\varphi_i^\tau(\langle i+1, j\rangle)$ computations for every j and every finite extension τ of o_i.

Look for the first (τ, j) such that it either *(i)* converges and is $\neq r_{i+1}(j)$, or *(ii)* for every extension of τ, it diverges.

$\mathfrak{b}(i+1)$ is then taken to be greater than [case *(i)*] the maximum between $\mathfrak{b}(i)$ and the greatest column reached during the computation, [case *(ii)*] the greatest column reached in the enumeration of the extensions of o_i; which is the column from which every extension will make the computation diverge on j.

The *lost* lemma

We need some tool to construct r

LEM

$\{r_i : i \in \omega\}$ s.t. $\forall i, r_i <_T r_{i+1}$

$\exists r$ s.t. $\forall i, r_i \leqslant_T r$ but $\bigoplus_i r_i \not\leqslant_T r \leqslant_T (\bigoplus_i r_i)'$

The *lost* lemma is actually very close to an old classical computability result

The *lost* lemma

8° Republica Argentina. *Publicaciones de la Comision nacional de la energia atomica. Miscelanea*, n° 1.

Il signale également un fascicule polycopié : *Contribution du Laboratoire d'astronomie de Lille*, n° 2. Numéro spécial, à l'occasion du Colloque international de Liége sur *Les particules solides dans les objets astronomiques*.

ARITHMÉTIQUE. — *Sur le semi-réseau constitué par les degrés d'indécidabilité récursive*. Note (*) de M. **Daniel Lacombe**, présentée par M. Émile Borel.

Extension de certains résultats de Kleene et Post (1) et solution de quelques questions posées par ces auteurs (2).

Nous utiliserons dans ce qui suit les définitions et les notations de S. C. Kleene-E. L. Post (1). Nous désignerons par $\mathbf{D}$ l'ensemble des *degrés d'indécidabilité* (ou, pour abréger : *degrés*) et par $\mathbf{D}_A$ l'ensemble des degrés arithmétiques ($\mathbf{D}_A \subset \mathbf{D}$). $\mathbf{D}$ et $\mathbf{D}_A$ sont munis d'une relation d'ordre partiel (notée $<$ et $\leqq$) et d'une opération $a \to a'$ partout définie (nous désignerons par $a^{(i)}$ le résultat de cette opération itérée i fois à partir de a).

Les deux théorèmes suivants se démontrent au moyen des méthodes classiques (fondées essentiellement sur la *forme normale de Kleene*) complétées par l'utilisation de fonctions majorantes. Les conditions (A) et (B) du théorème I constituent deux cas particuliers d'une condition plus générale que nous ne pouvons énoncer ici, et qu'il serait d'ailleurs intéressant d'élargir.

Théorème I. — *Soit* $S = u_0, u_1, \ldots, u_n, \ldots$ *une suite infinie de degrés, strictement croissante (c'est-à-dire telle que* $i < j$ *entraîne* $u_i < u_j$*) et satisfaisant à l'une ou l'autre des conditions suivantes :*

(A) *il existe un degré* a *tel que, pour tout* i, $u_i = a^{(i)}$;

(B) *il existe un degré* b *tel que, pour tout* i, $b < u_i < b'$.

Soit U *l'ensemble de degrés défini par la condition :*

$$x \in \mathrm{U} \Leftrightarrow \textit{il existe un } i \textit{ tel que } x \leqq u_i.$$

Soit d'autre part un degré c *n'appartenant pas à* U.

Dans ces conditions, on peut déterminer (d'une infinité de façons) deux degrés d_1 *et* d_2 *satisfaisant aux relations suivantes :*

(*) Séance du 27 octobre 1954.

(1) *Ann. Math.*, 59, 1954, p. 379-407.

(2) Ces questions sont en général indiquées, dans l'article cité ci-dessus, par le signe : [3], lequel renvoie à une Note de la page 380. Les résultats annoncés dans cette Note n'ont pas encore, à notre connaissance, été publiés.

(1) *pour tout degré* x, *on a*

$$(x \leqq d_1 \textit{ et } x \leqq d_2) \Leftrightarrow x \in \mathrm{U};$$

(2) *c ne vérifie aucune des deux inégalités*

$$c \leqq d_1 \quad \textit{et} \quad c \leqq d_2.$$

Remarque 1. — Ce théorème montre que S (ou, ce qui revient au même, U) ne possède pas de borne supérieure précise.

Remarque 2. — La relation (1) montre que le couple (d_1, d_2) ne possède pas de borne inférieure précise. De l'existence de suites S satisfaisant à (A) ou (B) on déduit donc immédiatement que $\mathbf{D}$ ne constitue pas un réseau [résultat démontré par Kleene et Post au moyen d'une suite de type (A)].

Remarque 3. — Kleene et Post ont montré l'existence de suites S satisfaisant à la condition (B), et cela pour n'importe quel degré b. Lorsque, dans cette hypothèse (B), le degré b appartient à $\mathbf{D}_A$, il en est de même pour b' et pour tous les u_i (et l'on a $\mathrm{U} \subset \mathbf{D}_A$). Cela n'entraîne pas forcément que d_1 et d_2 puissent être pris eux aussi dans $\mathbf{D}_A$. La relation (1) montre en effet que l'ensemble U est entièrement déterminé par la donnée de d_1 et d_2. Or $\mathbf{D}_A$ est dénombrable (donc aussi l'ensemble des couples formés de deux degrés arithmétiques). Mais Kleene et Post ont montré que les ensembles tels que U, déterminés dans $\mathbf{D}_A$ par des suites S de $\mathbf{D}_A$ croissantes et satisfaisant à (B) (avec b dans $\mathbf{D}_A$), forment une famille ayant la puissance du continu. Il en résulte que pour certaines de ces suites il n'existe aucun couple (d_1, d_2) formé de degrés arithmétiques et satisfaisant à la relation (1). Le théorème suivant donne une condition suffisante pour l'existence d'un tel couple.

Étant donnée une suite de degrés quelconque $S = u_0, u_1, \ldots, u_n, \ldots$ et une fonction φ de deux variables (entières $\geqq 0$), nous dirons que φ *énumère* S si, pour tout i, la fonction d'une variable φ_i définie par $\varphi_i(x) = \varphi(i, x)$ est de degré u_i.

Théorème II. — *Si, dans le théorème I-hypothèse (B), le degré b est arithmétique et si la suite S peut être énumérée par une fonction arithmétique, alors les degrés d_1 et d_2 satisfaisant aux relations (1) et (2) peuvent être pris (d'une infinité de façons) dans* $\mathbf{D}_A$.

Remarque 1. — Ce résultat reste valable quelque soit le degré c, arithmétique ou non.

Remarque 2. — Si S est énumérable par une fonction arithmétique, il en est de même pour l'ensemble U (qui est toujours dénombrable).

Remarque 3. — Les méthodes de Kleene-Post permettent de déterminer des suites S satisfaisant aux conditions de ce théorème II. Il en résulte que $\mathbf{D}_A$ ne constitue pas un réseau.

Compt. Rend. Ac. Sci. 239, 1954

The *lost* lemma

8° Republica Argentina. *Publicaciones de la Comision nacional de la energia atomica. Misce...* n° 1.

Il signale également un fascicule polycopié : *Contribution du Laboratoire d'astronomie de Lille*, n° 2. Numéro spécial à l'occasion du Colloque international de Liége sur *Les p...*

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(B) *il existe un degr...*

Soit U l'ensemble de ...

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Dans ces conditions, o... d_1 et d_2 satisfaisant aux ...

Daniel Lacombe

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Ohio State Basketball
Nationwide

ANNALS OF MATHEMATICS
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Printed in U.S.A.

ON DEGREES OF RECURSIVE UNSOLVABILITY*

BY CLIFFORD SPECTOR

(Received August 8, 1955)

In Kleene-Post [4][1] a number of questions concerning the structure of the upper semi-lattice of degrees were left unanswered. The present paper contains the answers to those questions under the scope of [4] Footnote 3. With the exception of the density problem ([4], 2.2), the methods used are variations of those developed in [4]. The construction employed in showing that the degrees are not dense involves a generalization of the methods of [4], and constitutes the main result of this paper (Theorem 4) that there are minimal degrees of recursive *un*solvability.[2] Familiarity with [4] is assumed.[3]

Annals of Mathematics, 64(3), 1956

CLIFFORD SPECTOR, MATHEMATICIAN, 30

Special to The New York Times.

PRINCETON, N. J., July 29—Dr. Clifford Spector, Associate Professor of Mathematics at the University of Michigan, died today of a cerebral hemorrhage at Princeton Hospital. He was 30 years old.

Dr. Spector, who had been doing research on a year's leave at the Institute of Defense Analysis here, lived at 327 Walnut Lane.

He previously had taught mathematics at the Ohio State University for five years and was an editor of the Journal of Symbolic Logic, a mathematics periodical.

Dr. Spector graduated from Columbia College, where he was elected to Phi Beta Kappa. He held a Master of Arts degree from Columbia and a doctorate from the University of Wisconsin.

In 1959 he was one of ten United States mathematicians to attend an international conference on symbolic logic at the Warsaw Academy of Science. Last year he did mathematical research at the Institute for Advanced Studies here.

He leaves his wife, the former Lea Eisner; two children, Alan and Judith; his parents, Mr. and Mrs. Robert Spector of New York, and a brother, Gilbert, Professor of Music at Kansas State College.

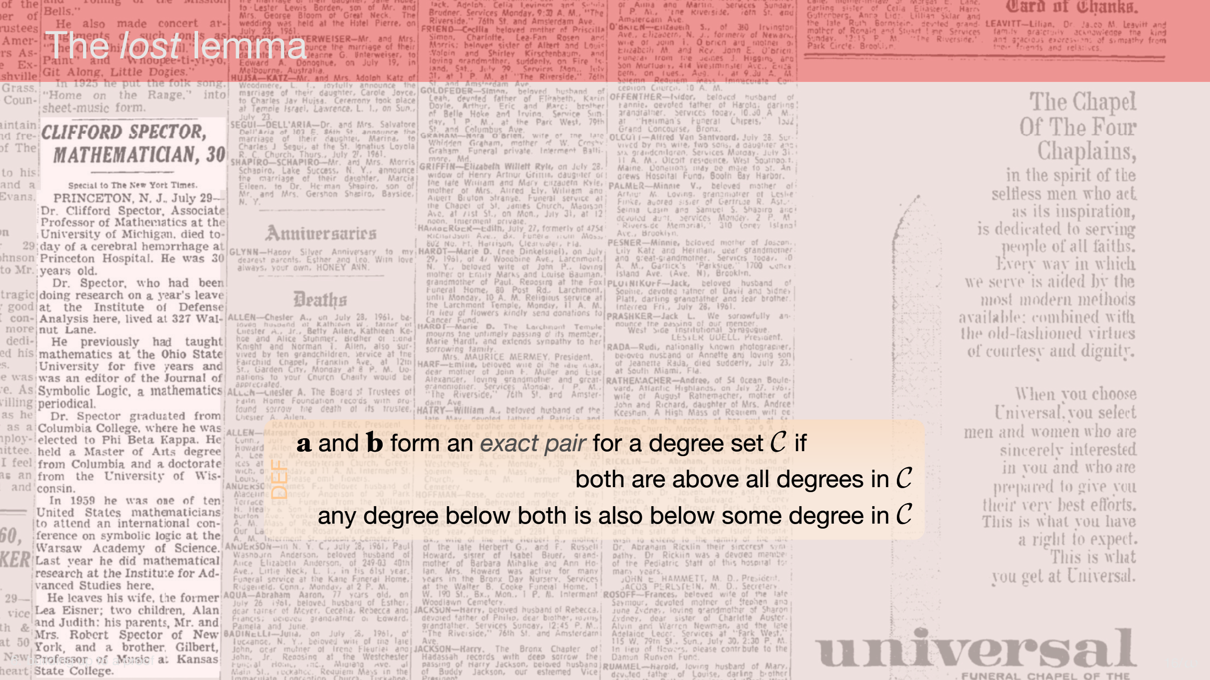
Anniversaries

Deaths

Card of Thanks

The New York Times, July 30, 1961

The *lost* lemma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Anniversaries

Deaths

DEF

a and **b** form an *exact pair* for a degree set $\mathcal{C}$ if

both are above all degrees in $\mathcal{C}$

any degree below both is also below some degree in $\mathcal{C}$

The *lost* lemma

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DEF $\mathbf{a}$ and $\mathbf{b}$ form an *exact pair* for a degree set $\mathcal{C}$ if

both are above all degrees in $\mathcal{C}$

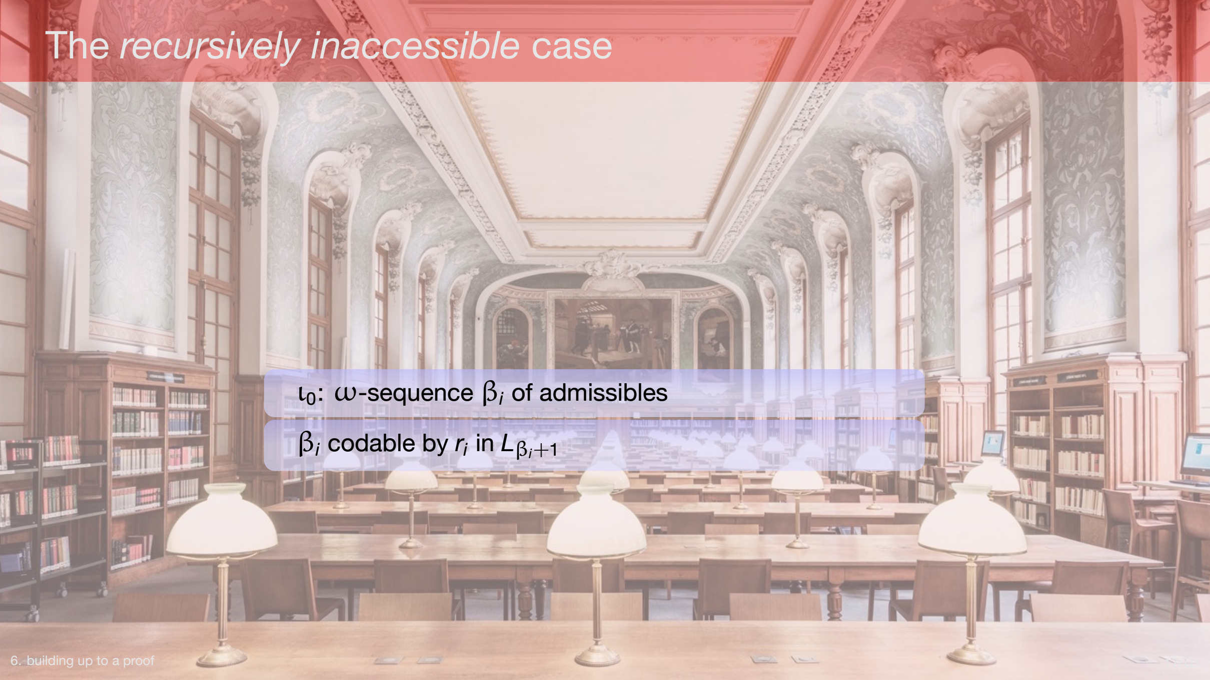
any degree below both is also below some degree in $\mathcal{C}$

THM Every countable set of degrees in which every pair of elements is bounded has an exact pair

The *recursively inaccessible* case

ι_0: ω-sequence β_i of admissibles

The *recursively inaccessible* case



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β_i codable by r_i in L_{β_i+1}

The *recursively inaccessible* case

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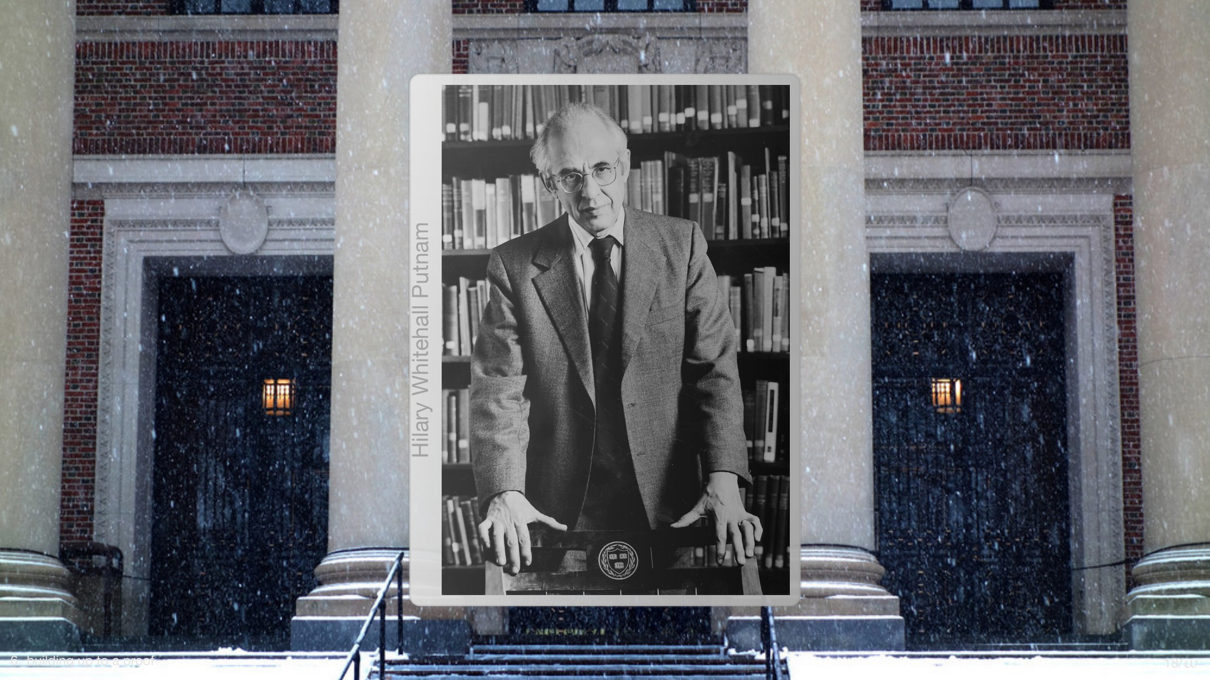
If the codes appear soon enough, same case than ι_0

The *recursively inaccessible* case

- ι_0: ω-sequence β_i of admissibles
- β_i codable by r_i in L_{β_i+1}
- *Lost* lemma: $\bigoplus_i r_i$ codes ι_0 and $\omega_1^{\mathrm{CK},r} = \iota_0$
- If the codes appear soon enough, same case than ι_0
- Where are the codes for the sequence members?


Harvard University


Hilary Whitehall Putnam

Putnam/definability gaps

THM There are arbitrarily long gaps in L where no news reals appear *Putnam* gaps

Putnam/definability gaps

THM There are arbitrarily long gaps in L where no news reals appear *Putnam* gaps

σχέδιο

Let $\beta > \alpha$ be countable ordinals such that there is an elementary embedding $j : L_\beta \to L_{\omega_2}$ with *critical point* $cr(j) \geqslant \alpha$.

For every $\gamma < cr(j)$,

$$L_{\omega_2} \models \text{“No new reals appear between ranks } \omega_1 \text{ and } \omega_1 + \gamma\text{.”}$$

No new reals thus appear between $cr(j)$ and $cr(j) + \gamma$, by elementarity and absoluteness.

Putnam/definability gaps

THM There are arbitrarily long gaps in L where no news reals appear *Putnam* gaps

LEM If new reals appear at $\alpha+1$, then among them is an arithmetical copy E_α of L_α

E_α is an *arithmetical copy* of L_α if there is one-one function f from L_α to ω (and onto the field of E_α) such that $\forall x, y \in L_\alpha$,

$$x \in y \iff \langle f(x), f(y)\rangle \in E_\alpha$$

Putnam/definability gaps

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Later seen as Jensen's mastercodes

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DEF r is a *master code* for $\mathcal{J}_\xi$ if

$$\{x \subseteq \omega : x \leqslant_T r\} = \mathcal{J}_{\xi+1} \cap \mathcal{P}(\omega).$$

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$\mathcal{J}_0 = \varnothing$, $\mathcal{J}_{\omega\cdot\xi} = L_\xi$, $\mathcal{J}_{\omega\cdot\xi+n} = \Delta_n(L_\xi)$, where λ is a limit ordinal and $\Delta_n(X)$ is the set of all subsets of X definable with parameters in $\langle X, \in \rangle$ by both Σ_n and Π_n first order formulae.

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THM: New reals appear at level ξ for $\mathcal{J}$ iff there is a master code for ξ. Furthermore, if r is a master code for ξ, then r' is the master code for $\xi+1$.

E_α is an *arithmetical copy* of L_α if there is one-one function f from L_α to ω (and onto the field of E_α) such that $\forall x, y \in L_\alpha$,

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THM There are countable ordinals α, υ such that α is not definable in L_υ

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$$x \in y \iff \langle f(x), f(y)\rangle \in E_\alpha$$

σχέδιος There exists α such that $L_\alpha \prec L_{\omega_1}$, α is thus not definable in L_{ω_1}.

There is a countable $\upsilon > \alpha$ such that $L_\upsilon \prec L_{\omega_1}$, and α is already not definable in L_υ.

New reals appear at level ξ for $\mathcal{J}$ iff there is a master code

THM for ξ. Furthermore, if r is a master code for ξ, then r' is the master code for $\xi+1$.

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Putnam/definability gaps

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Later seen as Jensen's mastercodes

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We can characterize the least such definability gap

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σχέδιος The ordinal υ_0, which is the least υ such that there is an ordinal α not definable in L_υ, can be characterized as the least η such that there exists an ordinal $\delta < \eta$ such that $L_\delta \prec L_\eta$.

$\{x \subseteq \omega : x \leq_T r\} = \mathcal{J}_{\xi+1} \cap \mathcal{P}(\omega)$.

$=$ $\Delta_n(L_\xi)$, where λ is a limit ordinal and $\Delta_n(X)$ is the set of all subsets of X definable with parameters in $\langle X, \in\rangle$ by both Σ_n and Π_n first order formulae.

THM New reals appear at level ξ for $\mathcal{J}$ iff there is a master code for ξ. Furthermore, if r is a master code for ξ, then r' is the master code for $\xi+1$.

THM There are countable ordinals α, υ such that α is not definable in L_υ

We can characterize the least such definability gap

Putnam/definability gaps

υ_0 is clearly $\leqslant$ the least such η, η_0, since whenever one has $L_\alpha \prec L_\beta$, α is not definable in L_β.

Now, suppose that $\upsilon_0 < \eta_0$, in other words, for all $\delta < \upsilon_0$, $L_\delta \not\prec L_{\upsilon_0}$. By Löwenheim-Skolem there is a countable elementary submodel of L_{υ_0}. Take the $\subseteq$-least such model M. By the Condensation Lemma, there is an $\alpha < \upsilon_0$ and an isomorphism j such that the Mostowski collapse of M is isomorphic to L_α via j. j cannot be trivial as this would mean that $L_\alpha \prec L_\delta$, although $\delta < \upsilon_0$ and υ_0 is the least such ordinal. We can thus consider κ, the critical point of j. Since $L_\alpha \cong M \prec L_{\upsilon_0}$, $L_\kappa \prec L_{j(\kappa)}$. But then κ cannot be definable in $L_{j(\kappa)}$, and thus $\upsilon_0 \leqslant j(\kappa)$. But $j(\kappa) < \upsilon_0$, contradiction.

We can characterize the least such definability gap

Definable/codable/countable

DEF

α *definable* at γ if definable without parameters in L_γ

α *codable* at γ if appears in $L_{\gamma+1}$ a real coding α

α *countable* at γ if $L_\gamma \models$ "α is countable".

Definable/codable/countable

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α *definable* at γ if definable without parameters in L_γ
α *codable* at γ if appears in $L_{\gamma+1}$ a real coding α
α *countable* at γ if $L_\gamma \models$ "α is countable".

$\alpha < \omega_1^L$ is codable at some level
if α is countable at β,
then α is codable and definable at β

Definable/codable/countable

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σχέδιος

Consider $\kappa = \aleph_\alpha$. κ is definable as the greatest cardinal in L_{κ^+}.

(Here κ^+ denotes the least ordinal of cardinality greater than κ.)

And thus α is also definable in L_{κ^+}.

Löwenheim-Skolem's theorem, in conjunction with Mostowski's lemma and the Condensation Lemma, provides the countable β such that α is definable in L_β.

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There is an upper bound for the ordinals that remain definable from some point on

Definable/codable/countable

α *definable* at γ if definable without parameters in L_γ

Memorable ordinals: ordinals α for which there exists β such that for any countable $\gamma \geqslant \beta$, α is still definable at γ.

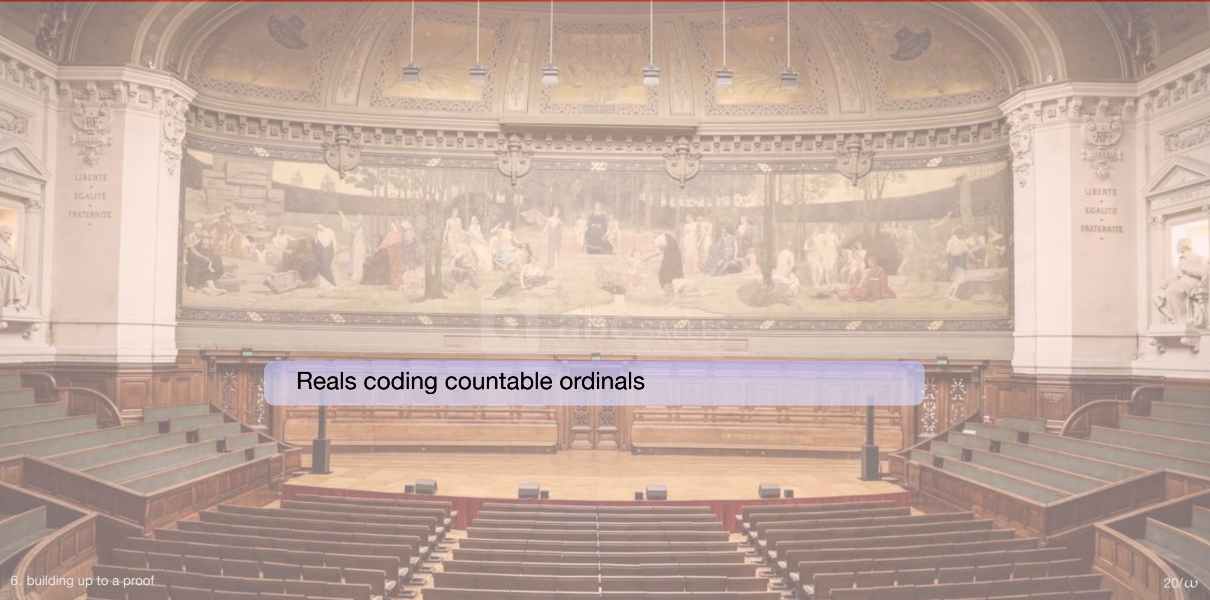
Any countable τ such that $L_\tau \prec L_{\omega_1}$ is such an upper bound: if α is definable at β, take δ above τ and β such that $L_\delta \prec L_{\omega_1}$. We then have $L_\tau \prec L_\delta \prec L_{\omega_1}$. α is thus definable at δ, since δ is above β, and also at τ. τ is therefore above α and any other definable ordinal. In fact, the least non-memorable ordinal τ_0 is the least ordinal τ with uncountably many elementary extensions $L_\tau \prec L_\gamma$.

σχέδιο

then not definable at a $\beta' > \beta$, *etc*

There is an upper bound for the ordinals that remain definable from some point on

Codability



Reals coding countable ordinals

Codability

Reals coding countable ordinals

LEM

Codes appear in an increasing way, sometimes in chunks

Codability

Reals coding countable ordinals

LEM Codes appear in an increasing way, sometimes in chunks

LEM If α and β are in the same codability gap and $\alpha \ll \beta$ then $r_\alpha <_T r_\beta$

The case of codability gaps

Successor admissible case

The case of codability gaps

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$\omega_1^{\mathrm{CK},r_\beta} \geqslant \alpha$ but $r_\alpha \not\leqslant_T r_\beta$, so $\omega_1^{\mathrm{CK},r_\beta} = \alpha$

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Recursively inaccessible case

Everything happens in L_γ

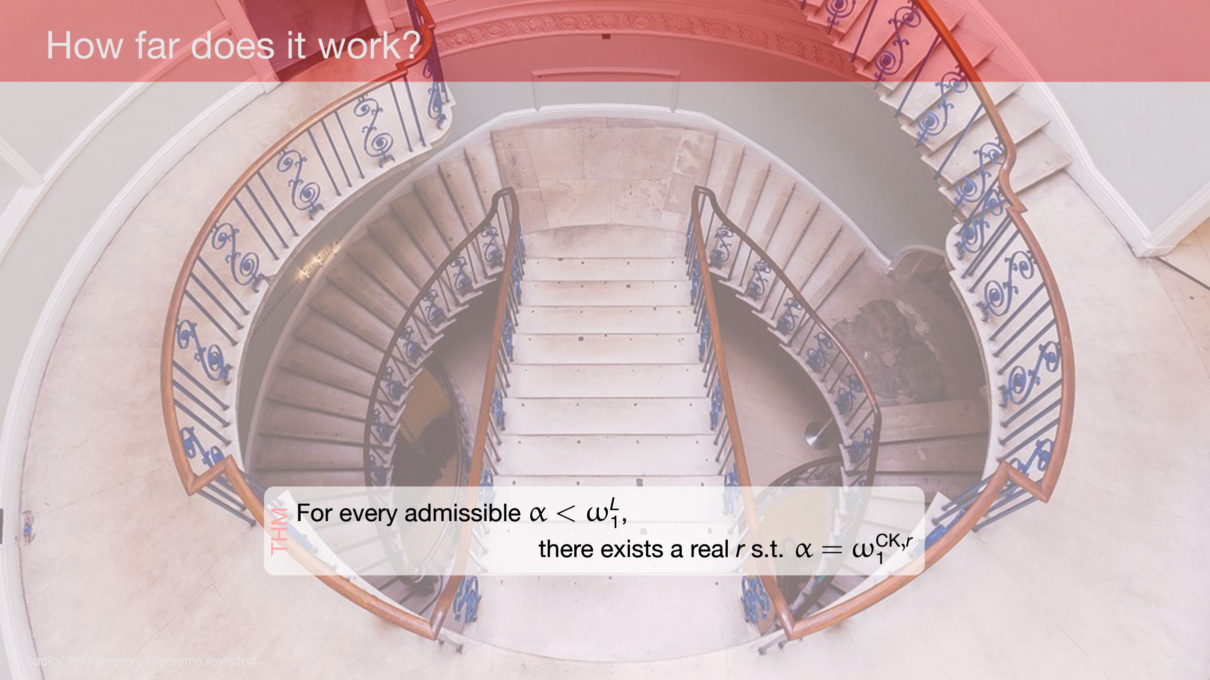
where the first code for α appears

7. sacks' and jensen's theorems revisited

Est-il possible de raisonner sur des objets qui ne peuvent être définis en un nombre fini de mots ? Est-il possible même d'en parler en sachant de quoi l'on parle, et en prononçant autre chose que des paroles vides ? Ou au contraire doit-on les regarder comme impensables ? Quant à moi je n'hésite pas à répondre que ce sont de purs néants.

—Poincaré, La logique de l'infini, 1909

How far does it work?



THM For every admissible $\alpha < \omega_1^L$,
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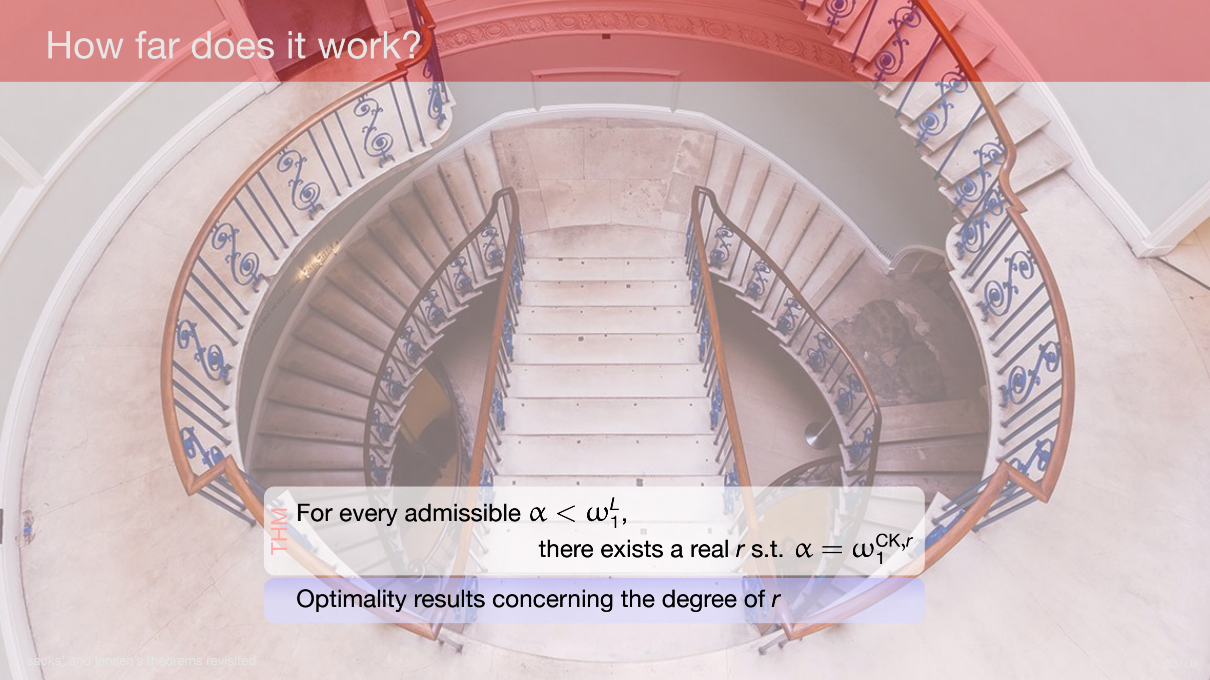

THM For every admissible $\alpha < \omega_1^L$,

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Optimality results concerning the degree of r


Universität Bonn

Ronald B. Jensen

Jensen's generalization

Sacks' theorem but for a sequence of admissibles?

Jensen's generalization

Sacks' theorem but for a sequence of admissibles?

But we have to be careful with the hypothesis

Jensen’s generalization

Sacks’ theorem but for a sequence of admissibles?

But we have to be careful with the hypothesis

σχέδιος

sequence of α admissibles

compatibility with the sequence of the first admissibles?

Jensen’s generalization

Sacks’ theorem but for a sequence of admissibles?

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DEF *Compatibility* hypothesis: having each member admissible relative to the initial segment

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THM $\langle \alpha_\beta : \beta < \gamma < \lambda_\infty \rangle$ sequence of admissibles $< \omega_1^L$ s.t. $\forall \delta < \gamma$, α_δ is admissible relative to $\{\alpha_\beta : \beta < \delta\}$

$\exists r$ s.t. α_β is the β-th r-admissible

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Use of infinite time Turing machines in generalized lemma

Jensen's generalization

Sacks' theorem but for a sequence of admissibles?

σχέδιο

$\beta + 1$-th in the sequence should not be decodable from r in L_{τ_β}

DEF

Compatibility hypothesis: having each member admissible relative to the initial segment

THM

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Use of infinite time Turing machines in generalized lemma


Thank you!